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A COLLECTION

OF

PROBLEMS AND EXAMPLES,

ADAPTED TO THE

"ELEMENTARY COURSE OF MATHEMATICS."

WITH

AN APPENDIX,

CONTAINING THE

QUESTIONS PROPOSED DURING THE FIRST THREE DAYS OF THE SENATE-HOUSE EXAMINATIONS IN THE YEARS 1848, 1849, 1850, AND 1851.

BY THE

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PREFACE TO THE FIRST EDITION.

THE following pages contain a collection of Problems and Examples adapted to my "Elementary Course of Mathematics." The questions have, for the most part, been collected from Papers which have been set in the Senate-House or in College Examinations; some have been taken from various Collections which have been published in Cambridge and elsewhere at different times; and the remainder I have myself supplied. I have compiled, rather than invented, as much as the circumstances of the case allowed, not only for the purpose of saving my own labour, but because the questions are more likely to be diverse in kind, and therefore more generally illustrative of the subject to which they belong, when supplied from a variety of sources.

The subjects upon which Problems and Examples will be found in this book are, Algebra, Trigonometry, Statics, Dynamics, Hydrostatics, and Optics. Of the three other subjects which are treated in my Course of Mathematics, namely, Conic Sections, the first three Sections of the *Principia*, and Astronomy, I have given no illustrations, for the following reasons. I considered that the geometrical method of treating the Conic Sections was re-introduced into the University principally, if not entirely, as an introduction to Newton's geometrical method of treating Mechanics, and that this end was answered if the student perfectly mastered

the fundamental propositions, without gaining such familiarity with the methods of demonstration as to enable him to apply the same or analogous methods to miscellaneous Problems. Moreover, the power of applying geometrical methods will be found to be possessed only by persons of great natural mathematical taste, and therefore such application cannot be expected from those who only study the elementary parts of Mathematics. Again, the three sections of Newton's Principia did not seem, for this last reason, to give rise to many Problems suitable to the class of students for whom this Collection is intended. And, lastly, the portions of Astronomy specified in the Grace of May, 1846, are so very limited as to make it difficult to frame any considerable number of illustrative Examples.

Very good Collections of Problems and Examples illustrative of several of the subjects above named are already in existence. I claim no superiority for the present, except its adaptation to the particular course of reading marked out by the Grace to which I have just now referred. All existing Collections, though they may contain questions adapted to the wants of a student whose reading is confined to an elementary course of Mathematics, such as that which my former work contains, have also, as might be expected, so great an admixture of Problems of a higher kind as extremely to trouble and perplex.

I have felt some doubt concerning the advantage of attaching answers to the questions proposed. In some instances such a course is manifestly undesirable, and

in many others it is doubtful. In the present Collection I have in general appended no answers, except to some of the algebraical questions, the answers to which are purely numerical; to work at a problem without seeing the result appears to me to be the more wholesome course for the student, partly because the form of the result may frequently give a hint concerning the Problem, and partly because the questions are so propounded in the Senate-House for which this Collection is only a preparation. I have however, I believe in all cases, myself worked and examined the Problems which I have admitted.

Certain of the questions require the aid of a table of logarithms for their solution; it will be understood therefore that the student is supposed to be in possession of such a table.

I cannot refrain from taking this opportunity of saying, that the extraordinary rapidity of the sale of the "Elementary Course" confirms me in the opinion I entertained of the necessity which existed for such a work, and at the same time gives me ground for hoping that my book, notwithstanding its defects, which are many, has in some measure answered the purposes for which it was written. I trust that the present little work will be found a useful supplement to the other.

H. GOODWIN.

CAMBRIDGE,
April 12, 1847.

PREFACE TO THE SECOND EDITION.

THE present is almost an exact reprint of the former edition. I have in fact been restrained from making some small improvements which might have been effected in the arrangement, from the necessity of making the collection still tally with Mr Hutt's volume of Solutions. In some instances in which the same problem had been by accident twice printed I have substituted new ones, and as a guide to the student who may be using the Examples in connection with the volume of Solutions these insertions have been distinguished by an asterisk.

I have given in the form of an Appendix the questions proposed in the Senate-House during the first three days of the Examinations of 1848, 1849, 1850, and 1851. This will, I have no doubt, be considered a useful addition.

H. G.

CAMBRIDGE,
October, 1851.

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ALGEBRA.

USE OF SYMBOLS.

1. FIND the value of $x^2 + \frac{x+1}{3}$, when $x = 8$.

2. Also of $x - \frac{\sqrt{x+2}}{3}$, when $x = 2$.

3. Also of $(2x + y)(2x + 3y)$, when $x = 2$, $y = 3$.

4. Also of $\frac{2x^2 + y^2}{x + 2y} - \frac{2x^2 - y^2}{x - 2y}$, when $x = 3$, $y = 1$.

5. Also of $(x + y + z)(x + y - z)(x + z - y)$, when $x = 1$, $y = 3$, $z = 5$.

6. Prove that $x - \{y - (z - u)\} = x - y + z - u$.

7. If $a = 1$, $b = 2$, $c = 3$, prove that

$$a^2 + b^2 + c^2 + 2ab + 2ac + 2bc = 6abc = a^3 + b^3 + c^3,$$

$$a^4 + b^4 + c^4 = 14(a + 2c),$$

$$a^5 + b^5 = 11c,$$

$$a^2 - ab + b^2 = c,$$

$$(a + b)(a + c)(b + c) = 10abc.$$

8. If $a = 2$, $b = 3$, $c = 5$, and $d = 6$, find the value of the following expressions:

$(2a + 3b)(4c - d)$, $(a + b + c - d)(a - 2c + 4d)$,

$(2b - 3d)(a + b - 3c + d)$, $a^2 - b^2 + c^2 - d^2$,

$(a^3 + b^3)(a^2b + b^2c + c^2d)$, and $(a + b + c + d)^3 - a^3 - b^3 - c^3 - d^3$.

9. Find the value of $x^3 - 3x^2 + 3x - 1$, when $x = 3$.

10. Find the value of $\sqrt{\frac{3}{4} - x} + \sqrt{2x} - \frac{3}{2}\sqrt{1 - 4x}$, when $x = \frac{1}{12}$.

11. Replace the radical signs in the following expressions by fractional indices:

$$\sqrt[4]{(x+y)^3},\ \sqrt{a^2x} + \sqrt[4]{b^3x^2} + \sqrt[6]{c^4x^3},\ \sqrt{\sqrt{\sqrt{a}}},\ \sqrt{a\sqrt{b\sqrt{c}}}.$$

ADDITION.

1. Add together $x^2 - ax + a^2$, $2x^2 + 3ax - 2a^2$, and $x^2 + ax + 3a^2$.

2. Add together $x^m + 4ax^{m-1} + a^2x^{m-2}$, $3x^m + 2a^2x^{m-2} + a^m$, and $ax^{m-1} + a^2x^{m-2} - 6a^m$.

3. Add together $2(a+b) - c + d$, $a + (b-c) - d$, and $a + b - (c-d)$.

4. Add together $a + 2c + d$, $2a - (b-c) - d$, and $3a + b - (3c + d)$.

5. Add together $a^3 + a^2b - ab^2 + b^3$, $a^3 - (a^2b + 3ab^2) - 2b^3$, and $2a^3 - a^2b + 2ab^2 + b^3$.

6. Add together $a(b+c-d)$, $b(a+d-c)$, $c(a+d-b)$, and $d(b+c-a)$.

7. Add together $a^2\{b-(c-d)\}$, $ab\{b+(a-c)\}$, and $b^2(c+d-a)$.

8. Add together $(3a+2b+c)x$, $(2a-3b-4c)x$, and $(a+b-c)x$.

9. Add together $ax - by$, $x - y$, and $(a-1)x + (b+1)y$.

10. Add together $2x^3 + 3(xy - y^2)$, $x^3 - y(2x + 3y)$, and $x(y - 3x) + y^2$.

11. Add together $(a+2b)x-(3a-4b)y, 2a(x+y)+b(x-y)$ and $4ax + 3by$.

12. Add together $a-\{b-(c-d)\}, a-b-(c-d), a+b-c+d$, and $a - (b + c - d)$.

13. Add together $2a^3+3a^2b-ac^2+c^3, a^3+2ac^2-b^3+3c^3$, $3a^2b + 4ab^2 - b^2c$, and $b^3 - 3ab^2 - 5a^2b - 4c^3$.

14. Add together $3a^4-4a^3b+ab^3+7a^2b^2, 2a^4+a^3b-6ab^3+b^4$, and $3a^3b + 5ab^3 - 6a^2b^2$.

15. Add together $a^2b^2c^2-a^4b^2+3ab^3c^2, abc^4+4a^2b^4+a^4b^2$, and $ab^3c^2 + 2a^2b^4 - b^6$.

16. Add together $a^2 - (ab + b^2)$, $2a^2 - (3ab-4b^2)$, and $4ab + 3b^2$.

17. Add together $17a^3 + 25a^2b - 71ab^2 + 28b^3$,
$161a^3 - 13a^2b + 14ab^2 - 51b^3$,
$19a^3 - 146a^2b + 317ab^2 - 80b^3$,
and $6a^3 - 23a^2b - 200ab^2 + 112b^3$.

18. Add together $121a^4 - 31a^3b + 60a^2b^2 + 28ab^3 - 40b^4$,
$30a^4 + 27a^3b - 15a^2b^2 + 37ab^3 + 12b^4$,
$17a^4 - 15a^3b - 40a^2b^2 - 52ab^3 + 7b^4$,
$a^4 - 3a^3b + 43a^2b^2 - ab^3 + 29b^4$,
and $6a^4 + a^3b - 29a^2b^2 + 8ab^3 - 16b^4$.

SUBTRACTION.

1. Subtract $a - b + c - d$, from $2a + b - c + 3d$.

2. Subtract $2a - 3(b + c)$, from $a - 2b + 2c$.

3. Subtract $a(x + y - z) + b(x - y + z)$, from $(a + b)x + (2a + 3b)y + (a - b)z$.

4. Subtract $a^2x + aby + 2b^2x$, from $3a(ax + by) - b^2x$.

5. Subtract $24a^2x - 17ax^2 + 5x^3$,
from $3a^3 + 25a^2x - 3ax^2 + 10x^3$.

6. Subtract $12(a + b)x - 13(a - b)y$,
from $20a(x + 2y) - 15b(2x - y)$.

7. Subtract $3a^{\frac{1}{3}}b^{\frac{1}{3}} + 4a^{\frac{2}{3}} - 5b^{\frac{2}{3}}$, from $3a^{\frac{2}{3}} + 9a^{\frac{1}{3}}b^{\frac{1}{3}}$.

8. Subtract $a + 2b + 2(a - b) - 3(a - 2b)$,
from $2a + 3(b - a) - 4(2b - a)$.

9. Subtract $ax^3 + bx^2y + cxy^2 + dy^3$,
from $dx^3 + cx^2y + bxy^2 + ay^3$.

10. Subtract $(a + b - c)x^3 + (b + c - d)x^2y$,
from $(a + c)x^2y - (a - b + c)y^3$.

11. Subtract $x(x^2 - xy + 3y^2) + y(2x^2 + 3xy - 2y^2)$,
from $x^3 + x^2y - 3xy^2 + 2y^3$.

12. Subtract $a - \{b - (c - d)\}$, from $2a + 3b - (2c - d)$.

13. Subtract $2a^3 - 3a^2b + ab^2 + b^3$, from $5a^3 + a^2b - 6ab^2 + b^3$.

14. Subtract $a^4 - 4a^3b + 6a^2b^2 - 7ab^3 + b^4$,
from $3a^4 - a^3b + 7a^2b^2 - 6ab^3 + b^4$.

15. Subtract $3a^5 + 5a^4b - a^3b^2 - 3ab^4 + 7b^5$,
from $3a^5 - 5a^4b + 2a^2b^3 + 9b^5$.

16. Subtract $7a^4 - 3(a^3b + ab^3) + b^4$,
from $9a^4 - 3a^3b - 7(ab^3 - b^4)$.

17. Subtract $a^2 + b^2 + c^2 - d^2$, from $a^2 - b^2 - c^2 + d^2$.

18. Subtract $256a^3 - 317a^2b + 19ab^2 - 36b^3$,
from $278a^3 - 311a^2b - 21ab^2 + 14b^3$.

MULTIPLICATION.

1. Multiply together a^2xy^2, ax^2y^3, and a^3x^3y.

2. Multiply together $a^{\frac{1}{3}}x^{\frac{2}{3}}$, $a^{\frac{2}{3}}x^{\frac{1}{3}}$, and $a^{\frac{1}{2}}x^{\frac{1}{2}}$.

3. Multiply together $a - bx + cx^2$, and $a + bx - cx^2$.

4. Multiply together $a - b + c$, $a + c - b$, and $a + b + c$.

5. Multiply together $x^6 - x^5y + x^4y^2 - x^3y^3 + x^2y^4 - xy^5 + y^6$,
and $x + y$.

6. Multiply together $2a^3b + 3a^2b^2 - ab^3 + 4b^4$,
and $a^4 - 4a^3b + a^2b^2 - 3ab^3$.

7. Multiply together $x^4 - 2x^3y + 4x^2y^2 - 8xy^3 + 16y^4$,
and $x^2 - 2y^2$.

8. Multiply together $x^2 - 2x + 3$, and $x^2 - x + 1$.

9. Multiply together $x^{\frac{2}{3}} - x^{\frac{1}{3}} + 2$, and $x^{\frac{2}{3}} + 3x^{\frac{1}{3}} - 1$.

10. Multiply together $x^{\frac{m}{m+n}} - x^{\frac{m-n}{m+n}}y^{\frac{m-n}{m+n}} + y^{\frac{n}{m+n}}$,
and $x^{\frac{n}{m+n}} + x^{\frac{m-n}{m+n}}y^{\frac{m-n}{m+n}} + y^{\frac{m}{m+n}}$.

11. Multiply together $a^2+b^2+c^2-ab-ac-bc$, and $a+b+c$.

12. Multiply together $a^4(b-c) + a^3(ab-bc)$,
and $a^3(ab+bc)$.

13. Multiply together $x - 1$, $x - 2$, and $x - 3$.

14. Multiply together $x^2 - 2x + 1$, $x^2 + 2x + 1$,
and $x^4 + 2x^2 + 1$.

15. Multiply together $x - a$, $x - b$, and $x - c$.

16. Multiply together $a + b$, $a - b$, $a^2 - ab + b^2$,
and $a^2 + ab + b^2$.

17. Multiply together $x + a$, $x + b$, $x + c$, and $x + d$.

18. Multiply together $1 + x + x^2 + \ldots\ldots + x^n$, and $1 - x$.

19. Multiply together $x^4 + 4x^3 + 6x^2 + 4x + 1$, and $x + 1$.

20. Multiply together $a^{(p-1)q} - b^{(q-1)p}$, and $a^q - b^p$.

21. Multiply together $a^{\frac{5}{2}} - 2a^2b^{\frac{1}{3}} + 4a^{\frac{3}{2}}b^{\frac{2}{3}} - 8ab + 16a^{\frac{1}{2}}b^{\frac{4}{3}} - 32b^{\frac{5}{3}}$, and $a^{\frac{1}{2}} + 2b^{\frac{1}{3}}$.

22. Multiply together $1 + \dfrac{x}{2} + \dfrac{x^2}{3} + \dfrac{x^3}{4}$, and $1 - \dfrac{x}{2} + \dfrac{x^2}{3} - \dfrac{x^3}{4}$.

23. Multiply together $a^{\frac{1}{2}} - a^{\frac{1}{4}}b^{\frac{1}{4}} + b^{\frac{1}{2}}$, and $a^{\frac{1}{2}} + a^{\frac{1}{4}}b^{\frac{1}{4}} + b^{\frac{1}{2}}$.

24. Multiply together $26a^{\frac{3}{2}} - 17a^{\frac{1}{2}}b + 6b^{\frac{3}{2}}$, and $7a^{\frac{1}{2}} - 2b^{\frac{1}{2}}$.

DIVISION.

1. Divide $a^5 + b^5$, by $a + b$.

2. Divide $a + b$, by $a^{\frac{1}{3}} + b^{\frac{1}{3}}$.

3. Divide $x^4 + x^3y + xy^3 + y^4$, by $x + y$.

4. Divide $x^5 - px^4 + qx^3 - qx^2 + px - 1$, by $x - 1$.

5. Divide $x(x-1)a^3 + (x^3 + 2x - 2)a^2 + (3x^2 - x^3)a - x^4$, by $a^2x + 2a - x^2$.

6. Divide $x^{\frac{3}{2}} - y^{\frac{3}{2}}$, by $x^{\frac{1}{2}} - y^{\frac{1}{2}}$.

7. Divide $x^{pq} - 1$, by $x^q - 1$.

8. Divide $\dfrac{x^4}{3} - \dfrac{11x^3}{12} + \dfrac{41x^2}{8} - \dfrac{23x}{4} + 6$, by $\dfrac{2x^2}{3} - \dfrac{5x}{6} + 1$.

9. Divide $x^4 - 9x^2 - 6xy - y^2$, by $x^2 + 3x + y$.

10. Divide $12a^4 - 26a^3b - 8a^2b^2 + 10ab^3 - 8b^4$, by $3a^2 - 2ab + b^2$.

11. Divide $a^8 + a^6b^2 + a^4b^4 + a^2b^6 + b^8$,
by $a^4 + a^3b + a^2b^2 + ab^3 + b^4$.

12. Divide $1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5$, by $1 - 3x + 3x^2 - x^3$.

13. Divide $a^4 + b^4 + c^4 - 2(a^2b^2 + a^2c^2 + b^2c^2)$, by $a^2 + 2ab + b^2 - c^2$.

14. Divide $a^6 + a^4b^2 - 2a^4c^2 - a^2b^4 + a^2c^4 - b^6 - 2b^4c^2 - b^2c^4$,
by $a^2 - b^2 - c^2$.

15. Divide $x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$,
by $x^2 + 2xy + y^2$.

16. Divide $x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$, by $x^3 + 3x^2y + 3xy^2 + y^3$.

17. Divide $x + 3x^{\frac{2}{3}}y^{\frac{1}{3}} + 3x^{\frac{1}{3}}y^{\frac{2}{3}} + y$, by $x^{\frac{1}{3}} + y^{\frac{1}{3}}$.

18. Divide $x^{\frac{4}{3}} - 4xy^{\frac{1}{3}} + 6x^{\frac{2}{3}}y^{\frac{2}{3}} - 4x^{\frac{1}{3}}y + y^{\frac{4}{3}}$, by $x^{\frac{2}{3}} - 2x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}$.

19. Divide 1 by $1 - x$ to four terms.

20. Divide $a - x$ by $a + x$ to six terms.

21. Divide $1 - x$ by $1 - x + x^2$ to four terms.

22. Divide $189a^4 - 91a^3b + 62a^2b^2 - 13ab^3 + 5b^4$,
by $27a^2 - 13ab + 5b^2$.

23. Divide $24a^4 - 10a^3b - 8a^2b^2 + 10ab^3 - 4b^4$,
by $4a^2 + ab - 2b^2$.

24. Is the following reasoning conclusive?

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots\ldots$$

Let $x = 1$;

$$\therefore\ \tfrac{1}{2} = 1 - 1 + 1 - 1 + \ldots\ldots \textit{ad infinitum.}$$

GREATEST COMMON MEASURE.

1. Find the Greatest Common Measure of $a^2 - b^2$, and $a^2 + 2ab + b^2$.

2. Of $a^3 + 3a^2b + 3ab^2 + b^3$, and $a^2 + 3ab + 2b^2$.

3. Of $a^3 + 2a^2b + 2ab^2 + b^3$, and $a^3 + b^3$.

4. Of $x^3 + x^2 - 2$, and $x^4 + x^3 - x - 1$.

5. Of $3x^3 - 3x^2y + xy^2 - y^3$, and $12x^2 - 15xy + 3y^2$.

6. Of $48x^2 + 16x - 15$, and $24x^3 - 22x^2 + 17x - 5$.

7. Of $3x^3 - 2x^2 - x$, and $6x^2 - x - 1$.

8. Of $6x^3 - 11x^2 + 5x - 3$, and $9x^3 - 9x^2 + 5x - 2$.

9. Of $x^3 - 8x^2 - 12x + 144$, and $3x^2 - 16x - 12$.

10. Of $x^4 + ax^3 - 9a^2x^2 + 11a^3x - 4a^4$, and $x^4 - ax^3 - 3a^2x^2 + 5a^3x - 2a^4$.

11. Of $x^4 - px^3 + (q - 1)x^2 + px - q$, and $x^4 - qx^3 + (p - 1)x^2 + qx - p$.

12. Of $3x^2 - (4a + 2b)x + 2ab + a^2$, and $x^3 - (2a + b)x^2 + (2ab + a^2)x - a^2b$.

13. Of $x^3 - 19x + 30$, and $x^3 - 2x^2 - 7x + 14$.

14. Of $9x^2 - 3xy - 6x + 2y$, and $6x^4 - 4x^3 - 3xy^2 + 2y^2$.

15. Of $48x^3 + 8x^2 + 31x + 15$, and $24x^3 + 22x^2 + 17x + 5$.

16. Of $x^6 + x^2y - x^4y^2 - y^3$, and $x^4 - x^2y - x^2y^2 + y^3$.

17. Of $a^3 + (a^2 + a)y + y^2$, and $a^4 - a^2(y^2 - y) - y^3$.

18. Of $x^4 - x^3 + x - 1$, and $x^4 - 2x^3 + 3x^2 - 2x + 1$.

19. Of $3x^5 - 10x^3 + 15x - 8$, and $x^4 - 2x^2 + 1$.

20. Of $15x^4 - 9x^3 + 47x^2 - 21x + 28$,
and $20x^6 - 12x^5 + 16x^4 - 15x^3 + 14x^2 - 15x + 4$.

21. Of $x^2 + 5x + 4$, $x^2 + 2x - 8$, and $x^2 + 7x + 12$.

22. Of $6x^3 + 4x^2y$, $2ax^4 - 8bx^2y^2$, and $4cx^5 + 12dx^4y$.

23. Of $a^3 + 5a^2x + 7ax^2 + 3x^3$, $ax^3 + 3a^2x - ax^2 - 3x^3$,
and $a^3 + a^2x - 5ax^2 + 3x^3$.

24. Of $x^4 - 2a(a-b)x^2 + (a^2+b^2)(a-b)x - a^2b^2$,
and $x^4 - (a-b)x^3 + (a-b)b^2x - b^4$.

25. Of $x^3 - 8x^2 - 12x + 144$, and $3x^2 - 16x - 12$.

26. Of $x^4 - x^3 - 3x^2 + 5x - 2$, and $x^5 - 2x^4 - x^3 + 5x^2 - 4x + 1$.

27. Of $20x^4 + x^2 - 1$, and $25x^4 + 5x^3 - x - 1$.

28. Of $nx^3 + 3nx^2y - 2nxy^2 - 2ny^3$,
and $2mxy^2 - 4mx^3 - mx^2y + 3my^3$.

29. Of $x^6 + 4x^5 - 3x^4 - 16x^3 + 11x^2 + 12x - 9$,
and $6x^5 + 20x^4 - 12x^3 - 48x^2 + 22x + 12$.

30. Of $e^{2x}a^3 + e^{2x} - a^3 - 1$, and $e^{2x}a^2 + 2e^xa^2 - 2e^x + a^2 - 2$.

LEAST COMMON MULTIPLE.

1. Find the Least Common Multiple of $x^2 - 1$,
and $(x+1)^2$.

2. Of $x^3 - 7x^2 + 16x - 12$, and $3x^2 - 14x + 16$.

3. Of $12x^2 - 17ax + 6a^2$, and $9x^2 + 6ax - 8a^2$.

4. Of $2x - 1$, $4x^2 - 1$, and $8x^3 + 1$.

5. Of $x^2 + 5x + 4$, $x^2 + 2x - 8$, and $x^2 + 7x + 12$.

6. Of $4(1-x)^2$, $8(1-x)$, and $8(1+x)$.

7. Of x^3-3x^2+3x-1, x^3-x^2-x+1, x^4-2x^3+2x-1, and $x^4-2x^3+2x^2-2x+1$.

8. Of $x^3-(4a+b)x^2+(3a^2+4ab)x-3a^2b$, and $x^3-(a+b)x^2-(30a^2-ab)x+30a^2b$.

9. Of $a^3+2a^2b-ab^2-2b^3$, and $a^3-2a^2b-ab^2+2b^3$.

10. Of $x^5+x^4+x^3+x^2+x+1$, and $x^5-x^4+x^3-x^2+x-1$.

11. Of $a^4-3a^3b+4a^2b^2-3ab^3+b^4$, and $a^4+a^3b+ab^3+b^4$.

12. Of $30a^2x^4-5a^3x^3+5a^5x$, and $9ax^3-a^3x+2a^4$.

13. Of $3x^3-2x^2-x$, and $6x^2-x-1$.

14. Of x^2-1, $(x-1)^2$, x^2+1, $(x+1)^2$, and x^3+1.

15. Of x^3-1, and x^2+x-2.

16. Of $6x^3-11x^2+5x-3$, and $9x^3-9x^2+5x-2$.

17. Of $21x^2-26x+8$, and $7x^3-4x^2-21x+12$.

18. Of x^2-1, x^2+2x-3, and x^3-7x^2+6x.

FRACTIONS.

Reduce the following to their simplest forms.

1. $\dfrac{1}{x-1}+\dfrac{1}{x+1}-\dfrac{2x}{x^2+1}$. Ans. $\dfrac{4x}{x^4-1}$.

2. $\dfrac{3x-2}{5}-\dfrac{x+7}{2}+4$. Ans. $\dfrac{x+1}{10}$.

3. $\dfrac{x-1}{2}+\dfrac{x-2}{3}+\dfrac{x+7}{6}$. Ans. x.

4. $\dfrac{x+\dfrac{1}{x}}{x^2+1}-\dfrac{1}{x+1}$. Ans. $\dfrac{1}{x(x+1)}$.

5. $\dfrac{a+\dfrac{b-a}{1+ba}}{1-a\dfrac{b-a}{1+ba}}$. Ans. b.

6. $\dfrac{a}{b}-\dfrac{(a^2-b^2)x}{b^2}+\dfrac{a(a^2-b^2)x^2}{b^2(b+ax)}$. Ans. $\dfrac{a+bx}{b+ax}$.

7. $\dfrac{1}{(a-b)(x+b)}+\dfrac{1}{(b-a)(x+a)}$. Ans. $\dfrac{1}{(x+a)(x+b)}$.

8. $\dfrac{a+x}{a-x}+\dfrac{a-x}{a+x}-2\dfrac{a^2-x^2}{a^2+x^2}$. Ans. $\dfrac{8a^2x^2}{a^4-x^4}$.

9. $\dfrac{x+y}{y}-\dfrac{2x}{x+y}+\dfrac{x^3-x^2y}{y^3-x^2y}$. Ans. $\dfrac{y}{x+y}$.

10. $\dfrac{1}{(a-b)(a-c)(x+a)}+\dfrac{1}{(b-a)(b-c)(x+b)}$

$+\dfrac{1}{(c-a)(c-b)(x+c)}$. Ans. $\dfrac{1}{(x+a)(x+b)(x+c)}$.

11. $\dfrac{\dfrac{1+\sqrt{5}}{2}x-2}{x^2-\dfrac{1+\sqrt{5}}{2}x+1}+\dfrac{\dfrac{1-\sqrt{5}}{2}x-2}{x^2-\dfrac{1-\sqrt{5}}{2}x+1}$.

Ans. $\dfrac{x^3-2x^2+3x-4}{x^4-x^3+x^2-x+1}$.

12. $\dfrac{1}{x-1}-\dfrac{1}{2x+2}-\dfrac{x+3}{2x^2+2}$. Ans. $\dfrac{x+3}{x^4-1}$.

13. $\dfrac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}}-\dfrac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}}$. Ans. $4x\sqrt{x^2-1}$.

14. $\dfrac{1}{2x+2} - \dfrac{4}{x+2} + \dfrac{9}{2(x+3)}$.

Ans. $\dfrac{x^2}{(x+1)(x+2)(x+3)}$.

15. $\dfrac{a}{(a-b)(a-c)(x-a)} + \dfrac{b}{(b-a)(b-c)(x-b)}$

$+ \dfrac{c}{(c-a)(c-b)(x-c)}$. Ans. $\dfrac{x}{(x-a)(x-b)(x-c)}$.

16. $\dfrac{a^2}{(a-b)(a-c)(x-a)} + \dfrac{b^2}{(b-a)(b-c)(x-b)}$

$+ \dfrac{c^2}{(c-a)(c-b)(x-c)}$. Ans. $\dfrac{x^2}{(x-a)(x-b)(x-c)}$.

17. $\dfrac{a^2+a+1}{(a-b)(a-c)(x-a)} + \dfrac{b^2+b+1}{(b-a)(b-c)(x+b)}$

$+ \dfrac{c^2+c+1}{(c-a)(c-b)(x-c)}$. Ans. $\dfrac{x^2+x+1}{(x-a)(x-b)(x-c)}$.

18. $\dfrac{1-x}{1+x} + \dfrac{(1-x)(1-x^2)}{(1+x)(1+x^2)}$. Ans. $\dfrac{2-2x}{1+x+x^2+x^3}$.

19. $\dfrac{a^2+ax+x^2}{a^3-a^2x+ax^2-x^3} \times \dfrac{a^2-ax+x^2}{a+x}$. Ans. $\dfrac{a^4+a^2x^2+x^4}{a^4-x^4}$.

20. $\dfrac{a^3-x^3}{a^3+x^3} \times \dfrac{a^2-x^2}{a^2+x^2} \times \dfrac{a-x}{a+x} \times \dfrac{a^2-ax+x^2}{a^2+ax+x^2}$.

Ans. $\dfrac{a^3-3a^2x+3ax^2-x^3}{a^3+a^2x+ax^2+x^3}$.

21. $\dfrac{x^2-9x+20}{x^2-6x} \times \dfrac{x^2-13x+42}{x^2-5x}$. Ans. $\dfrac{x^2-11x+28}{x^2}$.

22. $\dfrac{x^2+3x+2}{x^2+2x+1} \times \dfrac{x^2+5x+4}{x^2+7x+12}$. Ans. $\dfrac{x+2}{x+3}$.

23. $$\cfrac{x}{1+\cfrac{x}{1+x+\cfrac{x}{1+x+x^2}}}$$ Ans. $$\frac{x+3x^2+2x^3+x^4}{1+4x+3x^2+2x^3}.$$

24. $$\cfrac{1}{\cfrac{1}{x}+\cfrac{1}{x-1}+\cfrac{1}{x-2}}.$$ Ans. $$\frac{x^3-3x^2+2x}{3x^2-6x+2}.$$

INVOLUTION AND EVOLUTION.

1. Square $x^{\frac{1}{3}} - 1$, $x^{\frac{2}{3}} - x^{\frac{1}{3}} + 1$, and $x^2 - 2xy + y^2$.

2. Cube $ax^2 + bx + c$, $x^{\frac{4}{3}} - x + x^{\frac{2}{3}}$, and $x^5 - 1$.

3. Raise to the fourth power $x - 1$, $x^{\frac{1}{2}} - 2$, and $2x^{\frac{1}{4}} - 3$.

4. Write down the fifth powers of $-ax^2y$, $x^{\frac{1}{5}}y^{\frac{2}{5}}z^{\frac{1}{3}}$, and a^2xy^2.

5. Write down the m^{th} power of $a^{\frac{1}{2}}b^{\frac{2}{3}}x$, $a^{\frac{1}{4}}b^{\frac{1}{m}}x^{\frac{1}{3}}$, and ax^2y^3.

6. The cube of $(a-b)x + (a+b)y$,
is $(a-b)^3x^3 + 3(a-b)(a^2-b^2)x^2y + 3(a+b)(a^2-b^2)xy^2 + (a+b)^3y^3$.

7. Prove that $(a-x)^3 + (a+x)^3 = 2a^3 + 6ax^2$.

8. Also that $(1+x)^2 + (1+x)^3 + (1+x)^4$
$= 3 + 9x + 10x^2 + 5x^3 + x^4$.

9. $(a + bx + cx^2 + dx^3)^2 = a^2 + 2abx + (b^2 + 2ac)x^2$
$+ (2bc + 2ad)x^3 + (c^2 + 2bd)x^4 + 2cdx^5 + d^2x^6$.

10. $(1 + x + x^2 + x^3)^2 = 1 + 2x + 3x^2 + 4x^3 + 3x^4 + 2x^5 + x^6$.

11. Express $(1+x)^4 + (1-x+x^2)^2$ according to ascending powers of x.

12. Find the term involving x^3 in

$$(1 + x - x^2)^2 \times (1 - x + x^2)^3.$$

13. Write down the square roots of the following

$$a^{2m} x^{2n},\ a^{\frac{2}{p}} b^{\frac{4}{q}} c^{\frac{2}{r}},\ 64 a^4 x^2,\ 81 a^{\frac{2}{3}} x^{\frac{1}{3}},\ \text{and}\ 25 a^{\frac{2m}{m+n}} b^{\frac{2n}{m+n}}.$$

14. Write down the n^{th} root of $a^{mn} x^{np+n}$.

15. Extract the square root of $x^6 - 2x^4 + 2x^3 + x^2 - 2x + 1$.

16. Also of $x^2 - 2x^{\frac{3}{2}} + 3x - 2x^{\frac{1}{2}} + 1$.

17. Also of $x^4 - 4x^3 + 10x^2 - 12x + 9$.

18. Also of $x^6 - 2x^5 + 5x^4 + 2x^3 - 2x^2 + 12x + 9$.

19. Also of $4x^2y^4 - 12x^3y^3 + 17x^4y^2 - 12x^5y + 4x^6$.

20. Also of $9x^6 - 12x^5 + 10x^4 - 10x^3 + 5x^2 - 2x + 1$.

21. Also of $25a^6 - 30a^5x + 9a^4x^2 + 10a^3x^2 - 6a^2x^3 + x^4$.

22. Also of $x^2 + \dfrac{2ax}{3} - bx + \dfrac{a^2}{9} + \dfrac{b^2}{4} - \dfrac{ab}{3}$.

23. Also of $49x^4 - \dfrac{14x^3}{5} + \dfrac{1051}{25}x^2 - \dfrac{6x}{5} + 9$.

24. Also of $\dfrac{x^2}{y^2} + \dfrac{y^2}{x^2} - \left(\dfrac{x}{y} + \dfrac{y}{x}\right)\sqrt{2} + \dfrac{5}{2}$.

25. Extract the cube root of $x^3 - 3x^2 + 3x - 1$.

26. Also of $x^3 + 9x^2 + 27x + 27$.

27. Also of $8x^6 + 48x^5 + 60x^4 - 80x^3 - 90x^2 + 108x - 27$.

28. Also of $x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$.

29. Also of $\dfrac{a^3x^3}{b^3} - \dfrac{3a^2x^2}{b} + 3abx - b^3$.

30. Also of $x^6 + \frac{1}{x^6} - 6\left(x^4 + \frac{1}{x^4}\right) + 15\left(x^2 + \frac{1}{x^2}\right) - 20.$

31. Extract the fourth root of

$x^8 - 4x^7 + 10x^6 - 16x^5 + 19x^4 - 16x^3 + 10x^2 - 4x + 1.$

32. Also of $16x^8 + 32x^7 - 72x^6 - 136x^5 + 145x^4 + 204x^3 - 162x^2 - 108x + 81.$

33. Extract the fifth root of $x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1.$

34. Also of $x^{10} + 5x^9 + 15x^8 + 30x^7 + 45x^6 + 51x^5 + 45x^4 + 30x^3 + 15x^2 + 5x + 1.$

35. Prove that $\sqrt[3]{x^3 + 1} = x + \frac{1}{3x^2} - \frac{1}{9x^5} + \&c......$

36. Extract the seventh root of

$x^{14} + 14x^{13} + 91x^{12} + 364x^{11} + 1001x^{10} + 2002x^9 + 3003x^8 + 3432x^7 + 3003x^6 + 2002x^5 + 1001x^4 + 364x^3 + 91x^2 + 14x + 1.$

SIMPLE EQUATIONS.

Solve the following equations :

1. $x + \frac{2x}{3} = \frac{3x}{4} - 12.$ Ans. $x = -13\frac{1}{11}.$

2. $2x - 4 = 5x + 7.$ Ans. $x = -3\frac{2}{3}.$

3. $(x - 1)(x - 2) = (x - 3)(x - 4).$ Ans. $x = 2\frac{1}{2}.$

4. $\frac{a}{x} + \frac{b}{a} - \frac{a}{b} = 0.$ Ans. $x = \frac{a^2 b}{a^2 - b^2}.$

5. $\frac{9x + 7}{2} - \left(x - \frac{x - 2}{7}\right) = 36.$ Ans. $x = 9.$

6. $\dfrac{7x+4}{5} - x = \dfrac{3x-5}{2}$. Ans. $x = 3$.

7. $\dfrac{3}{1+\sqrt{x}} + \dfrac{3}{1-\sqrt{x}} = 4$. Ans. $x = -\frac{1}{2}$.

8. $\dfrac{3x}{4} + \dfrac{7x}{15} + \dfrac{11x}{6} - 366 = 0$. Ans. $x = 120$.

9. $\dfrac{3x+5}{8} - \dfrac{21+x}{3} = 39 - 5x$. Ans. $x = 9$.

10. $7x + 15 = 23 - \dfrac{1-9x}{2}$. Ans. $x = 3$.

11. $\dfrac{x-a}{3} - \dfrac{2x-3b}{5} - \dfrac{a-x}{2} = 10a + 11b$. Ans. $x = 25a + 24b$.

12. $\dfrac{3x+4}{5} - \dfrac{7x-3}{2} - \dfrac{x-16}{4} = 0$. Ans. $x = 2$.

13. $\dfrac{x-1}{7} + \dfrac{23-x}{5} = 7 - \dfrac{4+x}{4}$. Ans. $x = 8$.

14. $\dfrac{6x+a}{4x+b} = \dfrac{3x-b}{2x-a}$. Ans. $x = \dfrac{a^2-b^2}{b-4a}$.

15. $\dfrac{x}{8} - \dfrac{2x-2}{5} = \dfrac{3x-4}{15} + \dfrac{x}{12}$. Ans. $x = 1\frac{13}{67}$.

16. $\dfrac{x-2}{3} + \dfrac{x+3}{4} = 7 + \dfrac{x-4}{6}$. Ans. $x = 15$.

17. $\dfrac{x+1}{2} + \dfrac{x+2}{3} = 16 - \dfrac{x-6}{4}$. Ans. $x = 15\frac{1}{13}$.

18. $\dfrac{5x-7}{3} - \dfrac{4x-9}{5} + 2x = 13\frac{4}{5}$. Ans. $x = 5$.

19. $\dfrac{19}{3} - \dfrac{x-7}{3} = \dfrac{4x-2}{5}$. Ans. $x = 8$.

20. $\dfrac{5x-7}{3}-\dfrac{3x-2}{7}=\dfrac{x-5}{4}.$ Ans. $x=\dfrac{67}{83}.$

21. $\dfrac{5x}{9}-\dfrac{2x-1}{3}=\dfrac{4}{15}.$ Ans. $x=\dfrac{3}{5}.$

22. $a+x+\sqrt{a^2+bx+x^2}=b.$ Ans. $x=b\,\dfrac{b-2a}{3b-2a}.$

23. $\sqrt{4a+x}=2\sqrt{b+x}-\sqrt{x}.$ Ans. $x=\dfrac{(a-b)^2}{2a-b}.$

24. $\sqrt{1+x+x^2}+\sqrt{1-x+x^2}=\frac{1}{2}.$ Ans. $x=\dfrac{\sqrt{5}}{4}.$

25. $\sqrt{4a+x}+\sqrt{a+x}=2\sqrt{x-2a}.$ Ans. $x=\dfrac{17a}{8}.$

26. $\sqrt{x}+\sqrt{x-\sqrt{1-x}}=1.$ Ans. $x=\dfrac{16}{25}.$

27. $\dfrac{1+x^3}{(1+x)^3}+\dfrac{1-x^3}{(1-x)^3}=a.$ Ans. $x=\sqrt{\dfrac{a-2}{a+4}}.$

28. $\dfrac{1}{\sqrt{1-x}+1}+\dfrac{1}{\sqrt{1+x}-1}=\dfrac{1}{x}.$ Ans. $x=\dfrac{\sqrt{3}}{2}.$

29. $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}=b.$ Ans. $x=\dfrac{2ab}{1+b^2}.$

30. $\sqrt{a+x}=\sqrt{b}+\sqrt{x}.$ Ans. $x=\dfrac{(a-b)^2}{4b}.$

31. $\dfrac{x-1}{\sqrt{x}+1}=4+\dfrac{\sqrt{x}-1}{2}.$ Ans. $x=81.$

32. $.15x+.2-.875x+.375=.0625x-1.$ Ans. $x=2.$

33. $\sqrt{x+a}+\sqrt{x-a}=\dfrac{b}{\sqrt{x+a}}$. Ans. $x=-\dfrac{(a-b)^2+a^2}{2(a-b)}$.

34. $\dfrac{1}{\sqrt{x}}+\dfrac{1}{\sqrt{a}}=\sqrt{\dfrac{1}{a}+\sqrt{\dfrac{4}{ax}+\dfrac{9}{x^2}}}$. Ans. $x=4a$.

35. $\dfrac{\sqrt{a}-\sqrt{a-\sqrt{a^2-ax}}}{\sqrt{a}+\sqrt{a-\sqrt{a^2-ax}}}=b$. Ans. $x=a\left\{1-\left(\dfrac{2\sqrt{b}}{1+b}\right)^4\right\}$.

36. $\dfrac{\sqrt{1+x}-1}{\sqrt{1-x}+1}+\dfrac{\sqrt{1-x}+1}{\sqrt{1+x}-1}=a$. Ans. $x=\dfrac{4}{a^2}\sqrt{a^2-4}$.

QUADRATIC EQUATIONS.

1. $x^2-2x=0$. Ans. $x=0$, or 2.

2. $(x-1)(x-2)=1$. Ans. $x=\frac{1}{2}(3\pm\sqrt{5})$

3. $2x^2+5x-7=0$. Ans. $x=1$, or $-\frac{7}{2}$.

4. $2x=4+\dfrac{6}{x}$. Ans. $x=3$, or -1.

5. $\dfrac{x-1}{x-2}+\dfrac{x-2}{x-1}=1$. Ans. $x=\dfrac{3\pm\sqrt{-3}}{2}$.

6. $\dfrac{x^2}{2}-\dfrac{x}{3}+7\frac{3}{8}=8$. Ans. $x=\frac{3}{2}$, or $-\frac{5}{6}$.

7. $x^2=21+\sqrt{x^2-9}$. Ans. $x=\pm5$, or $\pm3\sqrt{2}$.

8. $(x^2+5)^2-4x^2=160$. Ans. $x=\pm3$, or $\pm\sqrt{-15}$.

9. $17x^2+19x-1848=0$. Ans. $x=-11$, or $9\frac{15}{17}$.

10. $x+\dfrac{24}{x-1}=3x-4$. Ans. $x=5$, or -2.

11. $8x + 1 + \frac{7}{x} = \frac{21 + 65x}{7} - 2.$ Ans. $x = \pm \frac{7}{3}.$

12 $\frac{2x - 10}{8 - x} - \frac{x + 8}{x - 2} = 2.$ Ans. $x = 7$, or $\frac{4}{5}$.

13. $\frac{x + \sqrt{x^2 - a^2}}{x - \sqrt{x^2 - a^2}} = \frac{x}{a}.$ Ans. $x = -\frac{a}{8}(3 \pm \sqrt{-7}).$

14. $\frac{x - 4}{\sqrt{x} + 2} = x - 8.$ Ans. $x = 9$, or 4.

15. $\frac{a + x}{\sqrt{a} + \sqrt{a + x}} = \frac{a - x}{\sqrt{a} - \sqrt{a + x}}.$ Ans. $x = \frac{a}{2}(1 \pm \sqrt{5}).$

16. $\sqrt{x^2 + \frac{1}{2}\sqrt{x^2 + 1664}} = x + 1.$ Ans. $x = 10$, or $-11\frac{1}{15}$.

17. $(x - c)\sqrt{ab} - (a - b)\sqrt{cx} = 0.$

Ans. $x = \frac{ac}{b}$, or $\frac{bc}{a}$.

18. $\frac{x}{x + 1} + \frac{x + 1}{x} = \frac{13}{6}.$ Ans. $x = 2$, or -3.

19. $x - \frac{x^3 - 8}{x^2 + 5} = 2.$ Ans. $x = 2$, or $\frac{1}{2}$.

20. $(x + 2)^2 = 6x + 2.$ Ans. $x = 1 \pm \sqrt{-1}.$

21. $\frac{\sqrt{1 + x}}{1 + \sqrt{1 + x}} = \frac{\sqrt{1 - x}}{1 - \sqrt{1 - x}}.$ Ans. $x = \pm \frac{\sqrt{3}}{2}.$

22. $x^2 - 34 = \frac{x}{3}.$ Ans. $x = 6$, or $-\frac{17}{3}$.

23. $x = \frac{10}{3} + \frac{x^2}{12}.$ Ans. $x = 6 \pm 2\sqrt{-1}.$

24. $x^2 - x + 5\sqrt{2x^2 - 5x + 6} = \dfrac{3x + 33}{2}$.

Ans. $x = 3$, or $-\frac{1}{2}$, or $\dfrac{5 \pm \sqrt{1329}}{4}$.

25. $\dfrac{x + 2}{x - 1} - \dfrac{4 - x}{2x} = \dfrac{7}{2}$. Ans. $x = 2$, or $-\frac{1}{2}$.

26. $\dfrac{x + 3}{2} + \dfrac{16 - 2x}{2x - 5} = 5\frac{1}{5}$. Ans. $x = 5$, or $6\frac{9}{10}$.

27. $\dfrac{2x}{x - 4} + \dfrac{2x - 5}{x - 3} + 1 = 0$. Ans. $x = 2$, or $3\frac{1}{5}$.

28. $5\dfrac{3x - 1}{1 + 5\sqrt{x}} + \dfrac{2}{\sqrt{x}} = 3\sqrt{x}$. Ans. $x = 4$, or $\frac{1}{9}$.

29. $x^{\frac{7}{3}} = 56\,x^{-\frac{2}{3}} + x^{\frac{5}{6}}$. Ans. $x = 4$, or $\sqrt[3]{49}$.

30. $a^2 b^2 x^{\frac{1}{n}} - 4\,a^{\frac{3}{2}} b^{\frac{3}{2}} x^{\frac{m+n}{2mn}} = (a - b)^2 x^{\frac{1}{m}}$.

Ans. $x = \left(\dfrac{1}{\sqrt{b}} \pm \dfrac{1}{\sqrt{a}}\right)^{\frac{4nm}{m-n}}$.

31. $(a + 1)(x^{\frac{1}{2}} - 1)^2 = (a - 1)(x + 1)$.

Ans. $x = \dfrac{a + 1}{2} \pm \dfrac{\sqrt{a^2 + 2a - 3}}{2}$.

32. $x^{-3} + \dfrac{1}{x\sqrt{x}} = 2$. Ans. $x = 1$, or $\sqrt[3]{\frac{1}{4}}$.

33. $4x^2 + 12x\sqrt{1 + x} = 27(1 + x)$.

Ans. $x = 3$, or $-\frac{3}{4}$, or $12\frac{1}{8}$, or $5\frac{5}{8}$.

34. $1 + \dfrac{x}{2} - \dfrac{x^2}{2\{1 + \sqrt{1 + x}\}^2} = 3$. Ans. $x = \infty, 0$, or 8.

SIMULTANEOUS EQUATIONS.

1. $\left.\begin{array}{l} 2x - 3y = 1 \\ 3x + y = 7 \end{array}\right\}.$ Ans. $x = 2$, $y = 1$.

2. $\left.\begin{array}{l} \dfrac{x}{3} + 2y = 5 \\ \dfrac{2x-1}{5} - y + 1 = 0 \end{array}\right\}$ Ans. $x = 3$, $y = 2$.

3. $\left.\begin{array}{l} x^2 + y^2 = a^2 \\ x + y = b \end{array}\right\}.$ Ans. $x = \frac{1}{2}\left\{b \pm \sqrt{2a^2 - b^2}\right\}$,
$y = \frac{1}{2}\left\{b \mp \sqrt{2a^2 - b^2}\right\}$.

4. $\left.\begin{array}{l} x^3 + y^3 = 9 \\ x + y = 3 \end{array}\right\}.$ Ans. $x = 1$, or 2.
$y = 2$, or 1.

5. $\left.\begin{array}{l} x^4 + y^4 = 272 \\ x - y = 2 \end{array}\right\}.$ Ans. $x = 4$, or -2, or $1 \pm \sqrt{-15}$,
$y = 2$, or -4, or $-1 \pm \sqrt{-15}$.

6. $\left.\begin{array}{l} x - \dfrac{y-2}{7} = 5 \\ 4y - \dfrac{x+10}{3} = 3 \end{array}\right\}.$ Ans. $x = 5$,
$y = 2$.

7. $\left.\begin{array}{l} 2x - \dfrac{y-3}{5} = 4 \\ 3y + \dfrac{x-2}{3} = 9 \end{array}\right\}.$ Ans. $x = 2$,
$y = 3$.

8. $$\left.\begin{array}{l}\sqrt{y}-\sqrt{a-x}=\sqrt{y-x}\\ \sqrt{y-x}+\sqrt{a-x}=\frac{5}{2}\sqrt{a-x}\end{array}\right\}.\qquad \text{Ans. } x=\tfrac{4}{5}a,\quad y=\tfrac{5}{4}a.$$

9. $$\left.\begin{array}{r}x+2y+3z=17\\ 2x+3y+z=12\\ 3x+y+2z=13\end{array}\right\}\qquad \text{Ans. } \left.\begin{array}{l}x=1\\ y=2\\ z=4\end{array}\right\}.$$

10. $$\left.\begin{array}{l}\left(\dfrac{x}{a}\right)^m\left(\dfrac{y}{b}\right)^n=c\\ \left(\dfrac{x}{b}\right)^n\left(\dfrac{y}{a}\right)^m=d\end{array}\right\}.\qquad \text{Ans. } \left.\begin{array}{l}x=c^{\frac{m}{m^2-n^2}}d^{\frac{n}{n^2-m^2}}a^{\frac{m}{m+n}}b^{\frac{n}{m+n}}\\ y=c^{\frac{n}{n^2-m^2}}d^{\frac{m}{m^2-n^2}}a^{\frac{m}{m+n}}b^{\frac{n}{m+n}}\end{array}\right\}.$$

11. $$\left.\begin{array}{r}2x+.4y=1.2\\ 3.4x-.02y=.01\end{array}\right\}.\qquad \text{Ans. } \left.\begin{array}{l}x=.02\\ y=2.9\end{array}\right\}.$$

12. $$\left.\begin{array}{l}x(bc-xy)=y(xy-ac)\\ xy(ay+bx-xy)=abc(x+y-c)\end{array}\right\}.$$

$$\text{Ans. } x^2=ac,\quad y^2=bc:$$

$$\text{or } x=\tfrac{1}{2}\left\{a-b+c\pm\sqrt{(a-b+c)^2-4ac}\right\},$$

$$y=\tfrac{1}{2}\left\{-a+b+c\mp\sqrt{(-a+b+c)^2-4bc}\right\}.$$

13. $$\left.\begin{array}{r}x+y+z=A\\ ax+by+cz=0\\ a^2x+b^2y+c^2z=0\end{array}\right\}.\qquad \text{Ans. } \left.\begin{array}{l}x=\dfrac{A}{\left(1-\dfrac{a}{b}\right)\left(1-\dfrac{a}{c}\right)}\\[2ex] y=\dfrac{A}{\left(1-\dfrac{b}{a}\right)\left(1-\dfrac{b}{c}\right)}\\[2ex] z=\dfrac{A}{\left(1-\dfrac{c}{a}\right)\left(1-\dfrac{c}{b}\right)}\end{array}\right\}.$$

14. $$\left.\begin{array}{r} x+y+z=A \\ (b+c)\,x+(c+a)\,y+(a+b)\,z=0 \\ bcx+\quad cay\quad+\quad abz\quad=0 \end{array}\right\}.$$

Ans. $$x=\frac{A}{\left(1-\frac{b}{a}\right)\left(1-\frac{c}{a}\right)},$$

$$y=\frac{A}{\left(1-\frac{a}{b}\right)\left(1-\frac{c}{b}\right)},$$

$$z=\frac{A}{\left(1-\frac{a}{c}\right)\left(1-\frac{b}{c}\right)}.$$

15. $$\left.\begin{array}{r} x^2-xy=2 \\ 2x^2+\ y^2=9 \end{array}\right\}.$$ Ans. $x=\pm\frac{1}{\sqrt{3}}$, or ± 2,

$y=\mp\frac{5}{\sqrt{3}}$, or ± 1.

16. $$\left.\begin{array}{r} xy=1225 \\ \sqrt{x}+\sqrt{y}=12 \end{array}\right\}.$$ Ans. $x=25$, or $107\pm 12\sqrt{71}$,

$y=49$, or $107\mp 12\sqrt{71}$.

17. $$\left.\begin{array}{r} xz=y^2 \\ x+y+z=14 \\ x^2+y^2+z^2=84 \end{array}\right\}.$$ Ans. $$\left.\begin{array}{r} x=2 \\ y=4 \\ z=8 \end{array}\right\}.$$

18. $$\left.\begin{array}{r} x+y+x^2+y^2=18 \\ xy=\ \ 6 \end{array}\right\}.$$ Ans. $$\left.\begin{array}{l} x=3,\text{ or } x=-3\pm\sqrt{3} \\ y=2,\ \ldots\ y=-3\mp\sqrt{3} \end{array}\right\}.$$

19. $$\left.\begin{array}{l} x^2+xy+y^2=a^2 \\ x+x^{\frac{1}{2}}y^{\frac{1}{2}}+y=b \end{array}\right\}.$$

Ans. $$x=\frac{1}{4b}\left\{a^2+b^2\pm\sqrt{(3a^2-b^2)(3b^2-a^2)}\right\},$$

$$y=\frac{1}{4b}\left\{a^2+b^2\mp\sqrt{(3a^2-b^2)(3b^2-a^2)}\right\}.$$

20. $$\left.\begin{aligned}\frac{x^2}{y^2}+\frac{y^2}{x^2}+\frac{x}{y}+\frac{y}{x}&=\frac{27}{4}\\ x-y&=2\end{aligned}\right\}.$$

Ans. $$\left.\begin{aligned}x&=4,\ \text{or}\ -2,\ \text{or}\ 1\pm\sqrt{\frac{3}{11}}\\ y&=2,\ \text{or}\ -4,\ \text{or}\ -1\pm\sqrt{\frac{3}{11}}\end{aligned}\right\}.$$

21. $$\left.\begin{aligned}yz&=a(y+z)\\ zx&=b(z+x)\\ xy&=c(x+y)\end{aligned}\right\}.$$

Ans. $$\left.\begin{aligned}x&=\frac{2}{\frac{1}{b}+\frac{1}{c}-\frac{1}{a}}\\ y&=\frac{2}{\frac{1}{a}+\frac{1}{c}-\frac{1}{b}}\\ z&=\frac{2}{\frac{1}{a}+\frac{1}{b}-\frac{1}{c}}\end{aligned}\right\}.$$

22. $$\left.\begin{aligned}xyz&=231\\ xyw&=420\\ yzw&=1540\\ xzw&=660\end{aligned}\right\}.$$ Ans. $x=3,\ y=7,\ z=11,\ w=20.$

23. $$\left.\begin{aligned}9x-2z+u&=41\\ 7y-5z-w&=12\\ 4y-3x+2u&=5\\ 3y-4u+3w&=7\\ 7z-5u&=11\end{aligned}\right\}.$$ Ans. $x=5,\ y=4,\ z=3,$ $u=2,\ w=1.$

24. $$\left.\begin{aligned}xyz&=105\\ \frac{x}{yz}&=\frac{3}{35}\\ \frac{xy}{z}&=\frac{15}{7}\end{aligned}\right\}.$$ Ans. $x=3,\ y=5,\ z=7.$

25. $$\left.\begin{array}{l} \dfrac{2x+z-4}{12}+\dfrac{3y-6z+1}{13}=\dfrac{x-2}{4} \\[2ex] \dfrac{3x-2y+5}{5}-\dfrac{4x-5y+7z}{7}=\dfrac{2}{7}+\dfrac{3y-9z+6}{6} \\[2ex] \dfrac{x}{9}-y+3z=2 \end{array}\right\}.$$

Ans. $x = 18,\ y = 12,\ z = 4.$

26. $$\left.\begin{array}{r} xy = (x+y) \\ xz = 2(x+z) \\ yz = 3(y+z) \end{array}\right\}.$$ Ans. $x = \dfrac{12}{7},\ y = \dfrac{12}{5},\ z = -12.$

27. $\sqrt{ax}+\sqrt{by}=\dfrac{x+y}{2}=a+b.$ Ans. $x = (\sqrt{a} \pm \sqrt{b})^2,$
$y = (\sqrt{a} \mp \sqrt{a})^2.$

28. $$\left.\begin{array}{r} xy^2z^3 = 108 \\ yz^2 = 18x \\ 2z = 3yx^2 \end{array}\right\}.$$ Ans. $x = 1,\ y = 2,\ z = 3.$

PROBLEMS PRODUCING EQUATIONS.

1. What number is that, from the double of which if 2 be subtracted, the remainder will be 10? Ans. 6.

2. There are two numbers, one of which is greater than the other by 3, and the difference between the greater and the double of the less is also 3: find the numbers.
Ans. 6 and 9.

3. A mercer bought 4 pieces of silk, which together measured 50 yards; the second was twice, the third three times, and the fourth four times as long as the first. What were the respective lengths of the pieces?
Ans. 5, 10, 15, 20 yards.

4. A gentleman buys four horses; for the second of which he gives £12 more than for the first; for the third £6 more than for the second; and for the fourth £2 more than for the third. The sum paid for all is £230. How much does each cost? Ans. £45. £57. £63. £65.

5. A man leaves his property, amounting to £26,000, among his children, namely, four sons and four daughters, in such a way that the eldest son is to receive half as much again as either of the younger, and a younger son's portion is to be twice that of a daughter. Determine the amount to be received by each.

Ans. The eldest receives £6000, a younger son £4000, and a daughter £2000.

6. Two workmen receive the same sum for their labour; but if one were to receive 15 shillings more and the other 9 shillings less, then one would receive just three times as much as the other. What do they receive?

Ans. 21 shillings each.

7. What number is that, the treble of which is as much above 40, as its half is below 51? Ans. 26.

8. A certain sum of money is to be raised upon two estates, one of which pays 19 shillings less than the other; and if 5 shillings be added to three times the less payment, it will be equal to twice the greater. What are the sums paid? Ans. 33 and 52 shillings.

9. A cistern is filled in twenty minutes by three pipes, one of which conveys ten gallons more, and the other five gallons less, than the third, per minute. The cistern holds 820 gallons. How much flows through each pipe in a minute? Ans. 22, 7, and 12 gallons respectively.

10. A and B began to play; A with exactly $\frac{4}{9}$ of the sum which B had. After winning £10, A had as much money as B. What had each at first?

Ans. A had £16, and B £36.

11. A courier, passing through a certain place, travels at the rate of 13 miles in 2 hours; 12 hours afterwards an-

other passes through the same place, travelling the same road, at the rate of 26 miles in three hours. How long and how far must the second travel before he overtakes the first? Ans. 36 hours, 312 miles.

12. A waterman finds that he can row with the tide from *A* to *B*, a distance of 18 miles, in an hour and a half, and that to return from *B* to *A* against the same tide, though he rows back along the shore where the stream is only three-fifths as strong as in the middle, takes him just two hours and a quarter. Find the rate at which the tide runs in the middle where it is strongest.

Ans. $2\frac{1}{2}$ miles per hour.

13. The hour and minute hand of a watch are together, and it is between four and five o'clock. Determine the precise time of day. Ans. $21\frac{9}{11}$ minutes past 4.

14. A packet sailing from Dover with a fair wind, arrives at Calais in two hours; on its return, the wind being contrary, it proceeds six miles an hour slower than it went. When it is halfway over, the wind changing, it sails two miles an hour faster, and reaches Dover sooner than it would have done, had not the wind changed, in the proportion of 6 : 7. Required the distance between Dover and Calais. Ans. 22 miles.

15. Fifteen guineas should weigh four ounces; but a parcel of light gold having been weighed and counted, was found to contain 9 more guineas than was supposed from the weight; and a part of the whole, exceeding the half by four guineas and a half, was found to be $1\frac{1}{2}$ oz. deficient in weight. What was the number of guineas. Ans. 36.

16. Two persons, *A* and *B*, played cards. After a certain number of games, *A* had won half as much as he had at first, and found that if he had 15 shillings more, he would have had just three times as much as *B*. But *B* afterwards won 10 shillings back, and he had then twice as much as *A*. What had each at first?

Ans. *A* had 14 and *B* 19 shillings.

17. What fraction is that, whose numerator being doubled, and denominator increased by 7, the value becomes $\frac{2}{3}$; but the denominator being doubled, and the numerator increased by 2, the value becomes $\frac{3}{5}$. Ans. $\frac{4}{5}$.

18. A winemerchant has two casks of wine, from the larger of which he draws 15 gallons, and from the smaller 11; the quantities remaining are in the proportion of 8 : 3. After they become half empty, he puts 10 gallons of water into each, and the quantities of liquor now in them are as 9 : 5. How many gallons will each hold?

Ans. They will hold 79 and 35 gallons respectively.

19. Two persons, A and B, can perform a piece of work together in 16 days. They work together for 4 days, when A being called off, B is left to finish it, which he does in 36 days more. In what time would each do it separately? Ans. A in 24 days, B in 48 days.

20. There is a cistern, into which water is admitted by three cocks, two of which are of exactly the same dimensions. When they are all open, five twelfths of the cistern is filled in four hours; and if one of the equal cocks is stopped, seven ninths of the cistern is filled in ten hours and forty minutes. In how many hours would each cock fill the cistern?

Ans. Each of the equal ones in 32 hours, and the other in 24.

21. Find two numbers in the proportion of 8 : 5, the product of which is 360. Ans. 24 and 15.

22. There are two numbers, whose sum is to their difference as 8 : 1, and the difference of their squares is 128. Find the numbers. Ans. 18 and 14.

23. The length of a certain rectangular field is to its breadth as 6 : 5. One sixth part of the area being planted, there remains for ploughing 625 square yards. What are the dimensions of the field?

Ans. The length is 30, and the breadth 25 yards.

24. A farmer bought two flocks of sheep, the first of which contained 18 fewer than the second. If he had given

for the first flock as many pounds as there were sheep in the second, and for the second as many pounds as there were sheep in the first, then the price of 6 sheep of the first flock would have been to the price of 7 sheep of the second in the proportion of 7 : 6. Required the number in each flock. Ans. 108 and 126.

25. There is a number consisting of two digits, which being multiplied by the digit on the left hand, the product is 46; and if the sum of the digits is multiplied by the same digit, the product is 10. Required the number.
Ans. 23.

26. Find two numbers, such that the product of the greater and the cube of the less may be to the product of the less and the cube of the greater as 4 : 9; and the sum of the cubes of the numbers may be 35. Ans. 3 and 2.

27. From two places, distant from each other 320 miles, two persons, *A* and *B*, set out at the same time to meet each other. *A* travelled 8 miles a day more than *B*, and the number of days in which they met was equal to half the number of miles *B* went in a day. How many miles did each travel per day? Ans. *A* 24 and *B* 16 miles.

28. A man travelled 105 miles, and then found that if he had not travelled so fast by 2 miles an hour, he should have been 6 hours longer in performing the same journey. Determine his rate of travelling. Ans. 7 miles per hour.

29. *A* and *B* entered into a speculation, to which *B* subscribed £15 more than *A*. After four months *C* was admitted, who added £50 to the stock; and at the end of 12 months from *C*'s admission they found they had gained £159; when *A* withdrawing received for principal and gain £88. What did he originally subscribe? Ans. £40.

30. There are three numbers, the difference of whose differences is 5; their sum is 20; and their continued product 130. Required the numbers. Ans. 2, 5, and 13.

31. A person bought a quantity of cloth of two sorts for £7 18*s*. For every yard of the better sort he gave as

many shillings as he had yards in all; and for every yard of the worse as many shillings as there were yards of the better sort more than of the worse. And the whole price of the better sort was to the whole price of the worse as 72 : 7. How many yards had he of each?

Ans. 9 of the better, and 7 of the worse.

32. There are two sorts of metal, each being a mixture of gold and silver, but in different proportions. Two coins of these metals of the same weight are to each other in value as 11 : 17; but if to the same quantities of silver as before in each mixture double the former quantities of gold had been added, the values of two coins of equal weights would have been to each other as 7 : 11. Determine the proportion of gold to silver in each mixture, the values of equal weights of gold and silver being as 13 : 1.

Ans. The proportion is 1 : 9 in the first mixture, and 1 : 4 in the second.

RATIOS, PROPORTION AND VARIATION.

1. Determine which is the greater ratio, $\frac{a+2b}{a+b}$ or $\frac{a+3b}{a+2b}$.

2. Compare the ratios, $\frac{\sqrt{3}}{\sqrt{2}}$ and $\frac{49}{40}$.

3. What quantity must be added to each of the terms of the ratio $a : b$, that it may become equal to the ratio $c : d$?

4. If four quantities are proportional, the sum of the greatest and least is greater than the sum of the other two.

5. If $a : b :: c : d$, and a be the greatest of the four quantities a, b, c, d, then $a^m + d^m$ is greater than $b^m + c^m$, m being any whole number.

6. If $a : b :: c : d$, then $a + b : a - b :: c + d : c - d$.

7. Given that $y \propto x$, and that when $x = 1$, $y = 3$, find the value of y when $x = 4$.

8. Given that $z \propto x + y$, that $y \propto x^2$, and that when $x = 1$, $y = 2$, and $z = 3$, determine the relation between z and x.

9. If z varies directly as y when x is constant, and inversely as x when y is constant, then when x and y both vary $z \propto \dfrac{y}{x}$.

10. If $z \propto x + y$, $u \propto x - y$, and $x \propto u + z$, then in general $y \propto x$. What exception is there to this conclusion?

11. If $z \propto y$, and $y \propto x$, then $y \propto \sqrt{xz}$.

12. If $z \propto y$, and $y \propto x$, then

$$x + y + z \propto \sqrt{yz} + \sqrt{xz} + \sqrt{xy}.$$

13. A sphere of metal is known to have a hollow space about its centre in the form of a concentric sphere, and its weight is $\frac{7}{8}$ of the weight of a solid sphere of the same substance and radius; compare the inner and outer radii, having given that the weight of a sphere varies as the cube of its radius.

14. There are two vessels, A and B, each containing a mixture of water and wine, A in the ratio of $2 : 3$, B in the ratio of $3 : 7$. What quantity must be taken from each, in order to form a third mixture which shall contain 5 gallons of water and 11 of wine?

ARITHMETICAL PROGRESSION.

1. The latter half of $2n$ terms of any arithmetical series $= \frac{1}{3}$ of the sum of $3n$ terms of the same series.

2. Sum the following series:

$2 + 5 + 8 + \ldots\ldots$ to 11 terms.

$3 + 7 + 11 + \ldots\ldots$ to 12 terms.

$m + m + 1 + m + 2 + \ldots\ldots$ to m terms.

$3 + 2 + 1 + \ldots\ldots$ to 8 terms.

$\frac{1}{2} + \frac{1}{3} + \frac{1}{6} + \ldots\ldots$ to 10 terms.

$\frac{1}{3} + \frac{5}{6} + \frac{4}{3} + \ldots\ldots$ to n terms.

$2\frac{1}{2} + 2 + \frac{3}{2} + \ldots\ldots$ to n terms.

$(a + x)^2 + (a^2 + x^2) + (a - x)^2 + \ldots\ldots$ to n terms.

$3 + 1 - 1 - \ldots\ldots$ to 10 terms.

$54 + 51 + 48 \ldots\ldots$ to 9 terms.

$6 + \frac{11}{2} + 5 + \ldots\ldots$ to 25 terms.

$\frac{n-1}{n} + \frac{n-2}{n} + \frac{n-3}{n} + \ldots\ldots$ to n terms.

3. The sum of an arithmetical series is 1455, the first term 5, the number of terms 30. What is the common difference?

4. Insert 3 arithmetical means between 2 and 14.

5. Given the n^{th} and m^{th} terms of an arithmetical series, required the sum of p terms.

6. $S_1, S_2, S_3 \ldots\ldots S_p$ are the sums of p arithmetical series continued to n terms; the first terms are 1, 2, 3,...... and the common differences 1, 3, 5,...... Prove that

$$S_1 + S_2 + S_3 + \ldots\ldots S_p = (np + 1)\frac{np}{2}.$$

7. If in the equation $s = \{2a + (n - 1)d\}\frac{n}{2}$, n has a negative value, prove that this value corresponds to a series of $-n$ terms, having the common difference d and $d - a$ for its first term.

8. There are n arithmetical means between 1 and 31, and the $7^{th} : \overline{n-1}|^{th} :: 5 : 9$; required the number of means.

9. Find three numbers in arithmetical progression, whose product = 120, and whose sum = 15.

10. Write down the arithmetical series, the 5^{th} and 9^{th} terms of which are respectively 1 and 9.

11. Determine the relation which must exist between a, b, and c, in order that they may be respectively the p^{th}, q^{th}, and r^{th} terms of an arithmetical progression.

12. The common difference of 4 numbers in arithmetical progression is 1, and their product 120; find the numbers.

13. Given the n^{th} term of an arithmetical series, and also the sum of n terms; find the series.

14. Prove that 1, 3, 5, 7...... is the only arithmetical progression beginning with unity, in which the sum of the first half of any even number of terms has to the sum of the second half the same constant ratio; and find that ratio.

15. The n^{th} term of an arithmetical series is $\frac{3n-1}{6}$; find the sum of n terms.

16. The sums of n terms of two arithmetical series are as $13 - 7n : 1 + 3n$; find the ratio of their first terms.

17*. Every square number can be represented under the form of an arithmetical progression commencing with unity.

18. In the two series 2, 5, 8,......and 3, 7, 11,......, each continued to 100 terms, how many terms are identical?

19. From two towns, 168 miles distant from each other, two persons, A and B, set out to meet each other; A goes 3 miles the first day, 5 the second, 7 the third, and so on; B goes 4 miles the first day, 6 the second, 8 the third, and so on; in how many days after starting will they meet?

GEOMETRICAL PROGRESSION.

1. Sum the following series:

$$1 + \frac{1}{2} + \frac{1}{4} + \ldots\ldots\ldots\ldots\ldots \text{ to 12 terms.}$$

$$2 + \frac{1}{5} + \frac{1}{50} + \quad \ldots\ldots\ldots\ldots \text{ to } n \text{ terms.}$$

$$\sqrt{\frac{3}{2}} + \sqrt{\frac{2}{3}} + \frac{2}{3}\sqrt{\frac{2}{3}} + \ldots \text{ to infinity.}$$

$$\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \ \ldots\ldots\ldots \text{ to infinity.}$$

$$1 - \frac{1}{5} + \frac{1}{25} - \ \ldots\ldots\ldots\ldots\ldots \text{ to 10 terms, and to infinity.}$$

$$\sqrt{\frac{3}{5}} - \sqrt{6} + 2\sqrt{15} - \ldots \text{ to 8 terms.}$$

$$\frac{2}{5} - \sqrt{\frac{2}{5}} + 1 - \&c\ldots\ldots\ldots \text{ to 8 terms.}$$

2. Find three numbers in geometrical progression, such that their sum = 14, and the sum of their squares = 84.

3. Prove that $.1111\ldots\ldots = \{.3333\ldots\ldots\}^2$.

4. The first term in a geometrical series is 1, and any one term of the series is equal to the sum of all that follow it if continued *ad infinitum;* required the series.

5. The arithmetical mean between a and b is greater than the geometrical.

6. If quantities are in geometrical progression, their differences are in geometrical progression.

7. There are three numbers in geometrical progression, whose sum = 13, and the sum of the first and second divided by the sum of the second and third = $\frac{1}{3}$. Required the numbers.

8. The difference between two numbers is 48, and the arithmetical mean exceeds the geometrical by 18. Find the numbers.

9. In any geometrical series, consisting of an even number of terms, the sum of the odd is to the sum of the even terms as $1 : r$, r being the common ratio.

10. Given the sum of three quantities in geometrical progression, and the sum of their reciprocals; find the quantities themselves.

11. In every geometrical progression consisting of an odd number of terms, the sum of the squares of the terms is equal to the sum of all the terms multiplied by the excess of the odd terms above the even.

12. The difference of the means of 4 numbers in geometrical progression is 2, and the difference of the extremes is 7; find the numbers.

13. The second and third terms of a geometrical progression are together equal to 24, and the next two to 216; what is the first?

14*. Find a geometrical series in which the sum of all the terms except the last is equal to the difference between the last and first terms.

15. If S_1, S_2,.........S_n be the sums of n geometrical series continued *ad infinitum*, the first term of which is 1, and the common ratios $\frac{1}{r}$, $\frac{1}{r^2}$,......$\frac{1}{r^n}$, respectively; find the value of the quantity

$$\frac{1}{S_1} + \frac{1}{S_2} + \ldots\ldots\ldots\ldots + \frac{1}{S_n}.$$

16. The $(p+q)^{\text{th}}$ term of a geometrical series = P, the $(p-q)^{\text{th}} = Q$; find the p^{th} and q^{th} terms.

17. If a, b, c, d are in geometrical progression, then

$$(a+b+c+d)^2 = (a+b)^2 + (c+d)^2 + 2(b+c)^2.$$

18. If P be the product, S the sum, and S' the sum of the reciprocals of n terms in geometrical progression, then

$$P = \left(\frac{S}{S'}\right)^{\frac{n}{2}}.$$

HARMONICAL PROGRESSION.

1. If the geometrical mean between x and y be to the harmonical as $1 : n$, prove that

$$\frac{x}{y} = \frac{1+\sqrt{1-n^2}}{1-\sqrt{1-n^2}}.$$

2. Continue in both directions the series 2, 3, 6.

3. In any harmonical progression, the product of the first two terms : the product of any two adjacent terms :: the difference between the first two : the difference between the two others.

4. The arithmetical mean between two numbers exceeds the geometrical by 13, and the geometrical exceeds the harmonical by 12, what are the numbers?

5. The arithmetical, geometrical, and harmonical means between a and b form a geometrical progression.

6. Insert 3 harmonical means between 1 and $\frac{1}{11}$.

7. If y be the harmonical mean between x and z, and x and z respectively the arithmetical and geometrical means between a and b, prove that

$$y = 2\frac{a+b}{\left\{\left(\frac{a}{b}\right)^{\frac{1}{4}} + \left(\frac{b}{a}\right)^{\frac{1}{4}}\right\}^2}.$$

8. Given the m^{th} and n^{th} terms of an harmonical progression; find the $(m+n)^{\text{th}}$ term.

9. There are four numbers, the first three of which are in arithmetical, and the last three in harmonical progression; it is required to prove that the first has to the second the same ratio which the third has to the fourth.

10. If P, Q, R be respectively the p^{th}, q^{th}, and r^{th} terms of an harmonical series, prove that

$$(p-q)PQ + (r-p)RP + (q-r)QR = 0.$$

PERMUTATIONS AND COMBINATIONS.

1. Find the number of permutations of 10 things taken 3 together, of 9 things taken 4 together, and of 6 things taken all together.

2. Find the number of permutations of 7 things taken all together, of which 3 are of one kind and 4 of another.

3. Find the number of permutations of the letters of the word "*science*," taken all together.

4. From a company of 50 men four are chosen every night to guard. On how many different nights can a different guard be posted; and on how many of these will any particular man be engaged?

5. How may triangles can be formed by joining the angular points of a polygon of n sides?

6. Find the number of combinations which can be formed by taking the letters of the alphabet 6 at a time, each combination containing two vowels and no more.

7. How many different arrangements can be made of 6 persons about a round table, such that a given person shall not have the same neighbours in any two arrangements?

8. The number of combinations of n things, taken p together, is to the number taken $p+2$ together as $a : b$. Find n. Example: $a = 5$, $b = 18$, $p = 3$.

9. The number of permutations of n things taken r together : the number of permutations taken $r-1$ together :: 10 : 1, and the corresponding combinations :: 5 : 3. Find the values of n and r.

10. At a game of cards, 3 being dealt to each person, any one can have 425 times as many hands as there are cards in the pack; required the number of cards.

11. How many words can be formed, consisting of 3 consonants and a vowel, in a language the alphabet of which consists of 19 consonants and 5 vowels?

12. Two dice have respectively n and $n+r$ faces; determine the numbers of different throws which can be made with them.

13. How many different hands is it possible to hold at whist?

14. Prove by the theory of combinations that the product of any n consecutive integers is divisible by $1 . 2 . 3 \ldots\ldots n$.

BINOMIAL THEOREM.

1. Expand the following:

$(1+x)^5$, $(a+b)^3$, $(a+bx+cx^2)^4$, $(1+x\sqrt{2})^6$, $(\sqrt{a}+\sqrt{x})^{10}$.

2. Expand the following, each to 5 terms,

$$(a+b)^{\frac{1}{2}},\ (1+x)^{\sqrt{2}},\ (1-x)^{\frac{1}{4}},\ (1-x)^{-\frac{1}{2}}.$$

3. The sum of the coefficients of $(1+x)^n$, n being a positive whole number, is 2^n; and the sum of the coefficients of the odd terms = the sum of the coefficient of the even = 2^{n-1}.

4. Find the sum of the squares of the coefficients of $(1+x)^n$, n being a positive integer.

5. Write down the term of the expansion of $\left(x+\frac{1}{x}\right)^n$, which does not involve x, n being an even number.

6. Write down the 4^{th} term of $(3a-2x)^{\frac{1}{2}}$, the 7^{th} term of $(2a-3x)^{\frac{1}{3}}$, and the 8^{th} term of $(1-x)^{-\frac{1}{3}}$.

7. The p^{th} coefficient of $(a+b)^n$ is greater than the $\overline{p+1}|^{\text{th}}$ if $p > \frac{n+1}{2}$, n being positive.

8. Given that the coefficient of the p^{th} term of the expansion of $(1+x)^n = P$, and that of the $(p+1)^{\text{th}} = P'$; find n.

9*. Find the greatest terms in the expansions of the following quantities;

$$(1+\tfrac{1}{3})^{18},\ (1-\tfrac{1}{2})^{\frac{9}{2}},\ (2-\tfrac{1}{3})^{-10}.$$

10. If $N =$ the n^{th} term of $(1-x)^m$, then the series after the first n terms

$$= Nx\left(1-\frac{m+1}{n}\right) + Nx^2\left(1-\frac{m+1}{n}\right)\left(1-\frac{m+1}{n+1}\right) + \&c.$$

11. If $z^2+z+1=0$, then will

$$\frac{1}{3}\left\{\left(1+\frac{x}{z}\right)^n + (1+x)^n + (1+xz)^n\right\}$$

be the sum of those terms of the expansion of $(1+x)^n$ in which the index of x is a multiple of 3.

12. The number of all possible combinations of n things is 2^n-1.

13. In the expansion of $\dfrac{1}{\sqrt{1 - x + x^2}}$, write down the coefficient of x^n.

14. Prove that four times the product of the sums of the odd and even terms of the expansion of $(a + b)^n$

$$= (a + b)^{2n} - (a - b)^{2n}.$$

15. Find an approximate value of $\sqrt{8}$, and of $\sqrt[3]{9}$, by the Binomial Theorem.

16. Find the sum of $1 + \dfrac{\alpha}{2} + \dfrac{\beta}{3} + \dfrac{\gamma}{4} + \ldots$, where $\alpha\beta\gamma\ldots$ are the coefficients of the expansion of $(1 + x)^n$.

17. The coefficient of the $\overline{r+1}|^{\text{th}}$ term of the expansion of $(1 + x)^{n+1}$ is equal to the sum of the coefficients of the r^{th} and $\overline{r + 1}|^{\text{th}}$ terms of $(1 + x)^n$.

18. Write down the p^{th} term of the expansion of $(1-x)^{\frac{1}{2}}$.

19. If a, b, c, d be consecutive coefficients of an expanded binomial, then

$$(bc + ad)(b - c) = 2(ac^2 - db^2).$$

20. Find the coefficient of x^n in $(1 + x + 2x^2 + 3x^3 + \ldots)^2$.

21. If $(1 + x)^n = a_0 + a_1x + a_2x^2 + \ldots$; find the value of the quantity $a_0a_1 + a_1a_2 + \ldots$, n being a positive integer.

22. Prove that

$$\left(\frac{1 + 2x}{1 + x}\right)^n = 1 + n\frac{x}{1 + 2x} + \frac{n(n + 1)}{1.2}\left(\frac{x}{1 + 2x}\right)^2 + \ldots\ldots$$

and that

$$\left(\frac{1 + x}{1 - x}\right)^n = 1 + n\frac{2x}{1 + x} + \frac{n(n + 1)}{1.2}\left(\frac{2x}{1 + x}\right)^2 + \ldots\ldots$$

23. If $\frac{x}{a} = 1 + h$, h being very small,

$$\frac{\left(2ax - x^2 + \frac{b^2x^2}{a^2}\right)^{\frac{1}{2}}}{(a^2 + b^2)^{\frac{1}{2}}} = 1 + \frac{b^2}{a^2 + b^2}h \text{ nearly.}$$

24. If x be very nearly equal to 1, prove that

$$\frac{a^p}{x^m} + \frac{b^q - a^p}{x^n} = b^q x^{(n-m)\frac{a^p}{b^q} - n} \text{ nearly.}$$

25. If x be very small compared with 1,

$$\frac{(1 + x)^{\frac{1}{2}} + (1 - x)^{\frac{2}{3}}}{1 + x + (1 + x)^{\frac{1}{2}}} = 1 - \frac{5}{6}x \text{ nearly.}$$

26. If $c = a - b$, and be very small compared with a and b, then

$$\frac{a^2b^2}{(a^2 - a^2x^2 + b^2x^2)^{\frac{3}{2}}} = a - 2c + 3cx^2 \text{ nearly.}$$

27. Given that the coefficient of the $(p + 1)^{\text{th}}$ term of the expansion of $(1 + x)^{2n}$ is equal to that of the $(p + 3)^{\text{th}}$ term, shew that $p = n - 1$.

28. Shew that the middle term of the expansion of

$$(1 + x)^{2n} = \frac{1 \cdot 3 \cdot 5 \ldots\ldots\ldots (2n - 1)}{1 \cdot 2 \cdot 3 \ldots\ldots\ldots n} 2^n x^n.$$

LOGARITHMS.

1. Solve the equations:

$$\left.\begin{aligned} a^x b^y &= c \\ a_1^x b_1^y &= c_1 \end{aligned}\right\}.$$

$$\left.\begin{aligned} a^x b^y c^z &= d \\ a_1^x b_1^y c_1^z &= d_1 \\ a_2^x b_2^y c_2^z &= d_2 \end{aligned}\right\}.$$

2. In a given country the births in one year amount to $\frac{1}{m}$th part of the whole population, and the deaths to $\frac{1}{n}$th part; in how many years will the population double itself?

3. A country trebles its population in a century; what is the increase in one year per million?

4. Prove that

$$\log 2 = 7 \log \frac{16}{15} + 5 \log \frac{25}{24} + 3 \log \frac{81}{80}.$$

5. If $20^x = 100$, $x = 1{\cdot}537$.

6. Apply the logarithmic tables to find the value of the following quantities:

$\sqrt[3]{2\sqrt[3]{4}}$. Ans. 1·7818.

$\sqrt[5]{7\sqrt{2}}$. Ans. 1·5817.

$\sqrt[6]{9\sqrt{3\sqrt{2}}}$. Ans. 1·6268.

7. If $a^{2x} - a^x = 1$, then $x = \dfrac{\log \frac{1}{2}(1 + \sqrt{5})}{\log a}$.

8. Given $a^1 a^3 a^5 \ldots\ldots = p$, find the number of factors $a^1, a^3, a^5 \ldots\ldots$

MISCELLANEOUS PROBLEMS.

1. Shew that $y^{mn} - 1$ is divisible by either of the quantities $y^m - 1$ or $y^n - 1$ without remainder, m and n being positive integers.

2. Of the two quantities $a^6 + a^4 b^2 + a^2 b^4 + b^6$ and $(a^3 + b^3)^2$, determine which is the greater.

3. Prove that

$$(a_1+a_2+\ldots+a_n)^2=n(a_1^2+a_2^2+\ldots+a_n^2)-(a_1-a_2)^2-(a_1-a_3)^2\ldots$$
$$-(a_2-a_3)^2\ldots\ldots\ldots\ldots\ldots$$

4. If a be greater than b, a^n-b^n is greater than $nb^{n-1}(a-b)$ and less than $na^{n-1}(a-b)$.

5. If a be an approximate value of the square root of n, and $n-a^2=\pm b$, then will

$$\sqrt{n}=a\pm\frac{2ab}{4a^2\pm b}, \text{ very nearly.}$$

6. If n be greater than 3, then will $\sqrt{n}$ be greater than $\sqrt[3]{n+1}$.

7. The number of different combinations of n things taken 1, 2, 3,...n at a time, of which there are p of one sort, q of another, and r of another,

$$=(p+1)(q+1)(r+1)-1.$$

8*. If $(b-a)(y-ma)=(nb-ma)(x-a)$,
and $(b'-a')(y-mb')=(mb'-na')(x-b')$,
then will

$$\left(\frac{1}{aa'}-\frac{1}{bb'}\right)x=\frac{a+a'}{aa'}-\frac{b+b'}{bb'}.$$

9. Prove that out of the combinations of n things the number of combinations involving an odd number of things exceeds the number of those involving an even by one.

10. Express $\dfrac{a+b\sqrt{-1}}{c+d\sqrt{-1}}$ in the form $A+B\sqrt{-1}$.

11. If $\dfrac{ad-bc}{a-b-c+d}=\dfrac{ac-bd}{a-b-d+c}$, then each of them

$$=\frac{a+b+c+d}{4}.$$

12. Reduce $\dfrac{\sqrt[3]{5.12}+\sqrt[3]{.03375}}{\sqrt[3]{80}-\sqrt{.01}}$ to its equivalent simple decimal.

13. If N and n be nearly equal, then

$$\sqrt{\frac{N}{n}} = \frac{N}{N+n} + \frac{1}{4}\frac{N+n}{n}, \text{ very nearly.}$$

14. Prove that the value of $x^2 - 2x + 3$ corresponding to $x = 1$ is smaller than for any other value of x.

15. Sum the series $\frac{a}{r} + \frac{a+b}{r^2} + \frac{a+2b}{r^3} + \ldots$ to n terms.

Ex. $\frac{1}{2} + \frac{3}{4} + \frac{5}{8} + \ldots\ldots$

16. Prove that $x^2 + y^2$ is never less than $2xy$.

17. Which is greater, $x - y$ or $(\sqrt{x} - \sqrt{y})^2$?

18. Shew that $\frac{a}{b^2} + \frac{b}{a^2}$ is greater than $\frac{1}{a} + \frac{1}{b}$.

19. Shew that $\sqrt{a^2+b^2}\sqrt{c^2+d^2}$ is greater than $ac+bd$.

20. Shew that abc is greater than $(a+b-c)(a+c-b)(b+c-a)$ unless $a = b = c$.

21. Shew that $a^3 + 1$ is greater than $a^2 + a$ unless $a = 1$.

22. Shew that $\frac{ma+nb}{m+n}$ is intermediate in value to a and b.

23. If $x = \frac{4ab}{a+b}$, then $\frac{x+2a}{x-2a} + \frac{x+2b}{x-2b} = 2$.

24. Prove that $x^4 + px^3 + qx^2 + rx + s$ is a perfect square, if $p^2 s = r^2$, and $q^2 = \frac{p^2}{4} + 2\sqrt{s}$.

25. If $ax^3 + bx^2 + cx + d$ is a perfect cube, prove that $ac^3 = db^3$.

26. Can x y and z be obtained from the following equations?

$$\begin{aligned} 3x - 2y + 5z &= 14, \\ 2x + \ \ y - 8z &= 10, \\ 8x - 3y + 2z &= 38. \end{aligned}$$

27. A quadratic equation cannot have more than two roots.

28. If $x_1 x_2$ are the roots of the equation $ax^2 + bx + c = 0$, prove that

$$\frac{x_1}{x_2} + \frac{x_2}{x_1} = \frac{b^2 - 2ac}{ac}.$$

29. The quantity $x^2 + ax + b$ is always positive, whatever be x, provided a^2 is less than $4b$.

30. The same value of x satisfies the equations

$$ax^2 + bx + c = 0,$$

and $a'x^2 + b'x + c' = 0$;

prove that $(ac' - a'c)^2 = (ab' - a'b)(bc' - b'c)$.

31. The same values of x and y will satisfy the three equations,

$$\begin{aligned} ax + by &= c, \\ a'x + b'y &= c'; \\ a''x + b''y &= c'', \end{aligned}$$

if $(a'b'' - a''b')c + (a''b - ab'')c' + (ab' - a'b)c'' = 0$.

32. If a, b, c be any quantities in geometrical progression,

$a^2 + b^2 + c^2$ is greater than $(a - b + c)^2$.

33. If $a^2 + b^2 + c^2 = 1$, and $a'^2 + b'^2 + c'^2 = 1$, then $aa' + bb' + cc'$ is never greater than 1.

34. If ${}_nP_r = \dfrac{n(n+1)\ldots\ldots(n+r-1)}{1\,.\,2\ldots\ldots r}$, prove that

$${}_nP_r + {}_{n+1}P_{r-1} = {}_{n+1}P_r.$$

TRIGONOMETRY.

GENERAL QUESTIONS CONCERNING TRIGONOMETRICAL FUNCTIONS.

1. Give the proper signs to $\sin\{(2n+1)180^\circ+\theta\}$ and $\cos\{(2n+1)180^\circ+\theta\}$, supposing θ to be less than 90°.

2. Trace the sign of the quantity $\sin\theta+\cos\theta$, while θ changes from 0 to 360°.

3. If θ be between 90° and 180°, what are the proper signs of $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$?

4. Write down a formula for all angles the tangent of which is $-\tan\theta$.

5. Write down a formula for all angles the cosine of which is $=1$.

6. Express all the trigonometrical functions in terms of the sine.

7. Express the same in terms of the tangent.

8. The same in terms of the versedsine.

9. The same in terms of the suversedsine, (or versedsine of the supplement.)

10. If $\tan\theta+\cot\theta=m$, express all the trigonometrical functions in terms of m.

11. Find $\sin\theta$ from the equation $\sin\theta\cos\theta=m$.

12. If the right angle were divided into 100° instead of 90°, what would be the value of an angle of $36^\circ\,7'$?

13. Determine the number of degrees into which the right angle must be divided in order that an angle of 30^0 may be measured by the number 40.

14. Shew that as the angle increases each of the trigonometrical functions changes sign whenever it passes through the value 0 or ∞, and for no other value.

FORMULÆ INVOLVING FUNCTIONS OF ONE ANGLE.

1. Prove the following formula:

$$\sin\theta = \sin\{(2n+1)\,180^0 - \theta\} = -\sin\{(2n+1)180^0 + \theta\}$$
$$= -\sin\{(2n+2)\,180^0 - \theta\},$$
$$\cos(360^0 - \theta) = -\cos(180^0 + \theta),$$
$$\sin(90^0 + \theta) = \sin(90^0 - \theta),$$
$$\tan\theta = -\tan(180^0 - \theta) = -\tan\{(2n+1)\,180^0 - \theta\},$$
$$= \tan(180^0 + \theta) = \tan\{(2n+1)\,180^0 + \theta\},$$
$$\operatorname{cosec}\theta = \operatorname{cosec}\{(2n+1)\,180^0 - \theta\},$$
$$\sec\theta = \sec(-\theta) = \sec(n\,.\,360^0 - \theta),$$
$$\sin\theta + \cos\theta + \tan\theta - \sec\theta = 0, \quad \text{when } \theta = 180^0,$$
$$\sin\theta\cot\theta = 1, \quad \text{when } \theta = 0,$$
$$\sin\theta - \cos\theta = 0, \quad \text{when } \theta = 45^0,$$
$$\sin\theta + \cos\theta = 0, \quad \text{when } \theta = 135^0.$$

2. If $\tan^3\theta = \dfrac{b}{a}$, then $\dfrac{a}{\cos\theta} + \dfrac{b}{\sin\theta} = (a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{3}{2}}$.

3. From the equation $\sin(a - \theta) = \cos(a + \theta)$, find θ.

4. Eliminate θ between the equations,

$$m = \operatorname{cosec}\theta - \sin\theta,$$
$$n = \sec\theta - \cos\theta.$$

5. Give a general formula for all values of θ which satisfy the equation

$$\cos\theta = -1.$$

6. Eliminate θ and ϕ from the equations,

$$\cos^2\theta = \frac{\cos\alpha}{\cos\beta}, \ \cos^2\phi = \frac{\cos\gamma}{\cos\beta}, \text{ and } \frac{\tan\theta}{\tan\phi} = \frac{\tan\alpha}{\tan\gamma}.$$

7. Eliminate θ and ϕ from the equations,

$$a\sin^2\theta + b\cos^2\theta = m,$$

$$b\sin^2\phi + a\cos^2\phi = n,$$

$$\text{and } a\tan\theta = b\tan\phi.$$

8. If $\tan\theta = \dfrac{b}{a}$, find the value of $a\cos\theta + b\sin\theta$.

FORMULÆ INVOLVING MORE THAN ONE ANGLE.

1. Prove the following formulæ:

$$\sin\theta = \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}},$$

$$\tan\theta = \cot\theta - 2\cot 2\theta,$$

$$\sin\theta + \cos\theta = \sqrt{1+\sin 2\theta},$$

$$\operatorname{cosec} 2\theta + \cot 2\theta = \cot\theta,$$

$$\cot\theta + \tan\theta = 2\operatorname{cosec} 2\theta,$$

$$2\operatorname{cosec} 2\theta = \sec\theta\operatorname{cosec}\theta,$$

$$\sec(\theta\pm\phi) = \frac{\sec\theta\sec\phi}{1\mp\tan\theta\tan\phi},$$

$$\cos\theta = \frac{1 - \tan^2\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}},$$

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta,$$

$$\tan(45^\circ + \theta) = \sqrt{\frac{1 + \sin 2\theta}{1 - \sin 2\theta}},$$

$$\tan^2\theta - \tan^2\phi = \frac{\sin(\theta - \phi)\sin(\theta + \phi)}{\cos^2\theta\ \cos^2\phi},$$

$$\frac{1 + \sin\theta}{1 + \cos\theta} = \frac{1}{2}\left(1 + \tan\frac{\theta}{2}\right)^2,$$

$$\frac{\sin 3\theta + \cos 3\theta}{\sin 3\theta - \cos 3\theta} = \frac{2\sin 2\theta + 1}{2\sin 2\theta - 1} \times \tan(45^\circ - \theta),$$

$$\sin n\theta = 2\sin(n - 1)\theta\cos\theta - \sin(n - 2)\theta,$$

$$\sin^2\theta - \sin^2\phi = \sin(\theta + \phi)\sin(\theta - \phi),$$

$$\sec^2\frac{\theta}{2} = \frac{2\sec\theta}{1 + \sec\theta},$$

$$\frac{\operatorname{vers}\theta}{\operatorname{vers}(180^\circ - \theta)} = \tan^2\frac{\theta}{2},$$

$$\frac{1 - 2\sin^2\theta}{1 + \sin 2\theta} = \frac{1 - \tan\theta}{1 + \tan\theta},$$

$$\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} = \tan 2\theta + \sec 2\theta,$$

$$\cos^2(\theta + \phi) = \sin^2\theta + \cos\phi\cos(2\theta + \phi),$$

$$\cot^2\left(45^\circ + \frac{\theta}{2}\right) = \frac{2\operatorname{cosec} 2\theta - \sec\theta}{2\operatorname{cosec} 2\theta + \sec\theta},$$

$$\sec\theta = \frac{2}{\sqrt{2 + \sqrt{2 + 2\cos 4\theta}}},$$

$$\frac{\sin\theta - \sin\phi}{\cos\theta - \cos\phi} + \cot\frac{\theta+\phi}{2} = 0,$$

$$\tan^2\frac{\theta}{2} = \frac{1-\cos\theta}{1+\cos\theta},$$

$$\frac{\cos\phi + \cos\theta}{\cos\phi - \cos\theta} = \cot\frac{\theta+\phi}{2}\cot\frac{\theta-\phi}{2},$$

$$2(\sin^2\theta\sin^2\phi + \cos^2\theta\cos^2\phi) = 1 + \cos 2\theta\cos 2\phi,$$

$$\cos(A+B)\sin(A-B) + \cos(B+C)\sin(B-C)$$
$$+ \cos(C+D)\sin(C-D) + \cos(D+A)\sin(D-A) = 0,$$

$$\tan^2\frac{\theta}{2} = \frac{2\sin\theta - \sin 2\theta}{2\sin\theta + \sin 2\theta},$$

$$\text{vers}\,(180^0 - \theta) = 2\,\text{vers}\left(90^0 + \frac{\theta}{2}\right)\text{vers}\left(90^0 - \frac{\theta}{2}\right),$$

$$\sin(\theta+\phi) - \sin\theta = \sin\theta - \sin(\theta-\phi) - 4\sin\theta\sin^2\frac{\phi}{2},$$

$$4\sin(\theta-\phi)\sin(m\theta-\phi)\cos(\theta-m\theta) = 1 + \cos(2\theta - 2m\theta)$$
$$-\cos(2\theta-2\phi) - \cos(2m\theta - 2\phi).$$

2. If $\dfrac{m\tan(a-\theta)}{\cos^2\theta} = \dfrac{n\tan\theta}{\cos^2(a-\theta)}$,

then $\tan(a-2\theta) = \dfrac{n-m}{n+m}\tan a.$

3. Find $\tan\theta$ from the equation

$$\frac{m}{n} = \frac{\sin a\cos(\beta+\theta)}{\sin\beta\cos(a-\theta)}.$$

4. If $\tan\theta = \sin 2\theta$, find the value of chd θ.

5. If A, B, C, be three angles the values of which form an arithmetical progression, then

$$\sin A - \sin C : \cos C - \cos A :: \cos B : \sin B.$$

6. If $\tan\dfrac{a}{2} = \tan^3\dfrac{\beta}{2}$, and $\tan\beta = 2\tan\gamma$, then γ is an arithmetical mean between a and β.

7. If $\tan\theta = \tan^3\dfrac{\phi}{2}$, and $3\cos^2\phi = m^2 - 1$, then

$$\cos^{\frac{2}{3}}\theta + \sin^{\frac{2}{3}}\theta = \left(\frac{2}{m}\right)^{\frac{2}{3}}.$$

8. Given $\sin\theta = m\sin\phi$, and $\tan\theta = n\tan\phi$, find $\sin\theta$ and $\cos\phi$.

9. Given $\sin(\theta + a) = m\sin\theta$, find θ in terms adapted to logarithmic computation.

10. If $\tan\phi = \dfrac{2\sin\theta\sin\psi}{\sin(\theta+\psi)}$, then $\tan\theta$, $\tan\phi$, $\tan\psi$ are in harmonical progression.

11. If $\theta + \phi + \psi = (2n+1)90^\circ$,

$$\tan\phi\tan\psi + \tan\psi\tan\theta + \tan\theta\tan\phi = 1,$$

and $4\cos\theta\cos\phi\cos\psi = \sin 2\theta + \sin 2\phi + \sin 2\psi$.

12. The tangents of two angles are $\frac{1}{2}$ and $\frac{1}{3}$ respectively, find the tangent of their sum.

13. Find θ from the equation

$$\tan\frac{\theta}{2} = \operatorname{cosec}\theta - \sin\theta.$$

14. If $\cot x = n\cot(a - x)$, then

$$\sin(a - 2x) = \frac{n-1}{n+1}\sin a.$$

15. Express $\cos 5\theta$ in terms of $\cos\theta$ and its powers.

16. Express $\tan 3\theta$ in terms of $\tan\theta$.

17. Determine θ from the equation

$$\sin a + \sin(\theta - a) + \sin(2\theta + a) = \sin(\theta + a) + \sin(2\theta - a).$$

18. Find the values of θ which satisfy the equation

$$2\sin^2 3\theta + \sin^2 6\theta = 2.$$

19. Find θ from the following,

$$\tan\theta + 2\cot 2\theta = \sin\theta\left(1 + \tan\theta\tan\frac{\theta}{2}\right).$$

20. If $\cos(\theta - \psi)\cos\phi = \cos(\theta - \phi + \psi)$, then $\tan\theta$, $\tan\phi$, $\tan\psi$ are in harmonical progression.

21. If $1 = \dfrac{\sin^2\phi}{\sin^2\theta} + \cos^2\phi\cos^2\psi$, then $\sin\psi = \dfrac{\tan\phi}{\tan\theta}$.

22. If $\cos\theta = \cos\phi\cos\psi$, then

$$\tan\frac{\theta+\phi}{2}\tan\frac{\theta-\phi}{2} = \tan^2\frac{\psi}{2}.$$

23. If $\sin\phi = m\sin(2\theta + \phi)$, then

$$\tan(\theta + \phi) = \frac{1+m}{1-m}\tan\theta.$$

24. Find θ from the following equation,

$$\cos\theta + \cos 2\theta + \cos 3\theta = 0.$$

NUMERICAL VALUES OF FUNCTIONS.

1. Assuming that $\cos 36^\circ = \dfrac{\sqrt{5}+1}{4}$, prove that

$$\sin\theta + \sin(36^\circ - \theta) + \sin(72^\circ + \theta) = \sin(36^\circ + \theta) + \sin(72^\circ - \theta).$$

2. Prove that $\sin 15^\circ = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$, and thence deduce the value of $\cos 15^\circ$, chd 15°, and $\tan 15^\circ$.

3. $\text{Sin } 75^\circ = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$.

4. Prove that

$$\sin 18^\circ : \sin 30^\circ :: \sin 30^\circ : \sin 54^\circ.$$

5. $2\cos 11^\circ 15' = \sqrt{2 + \sqrt{2 + \sqrt{2}}}$.

6. If $\tan\theta = \frac{1}{7}$ and $\tan\phi = \frac{1}{3}$, then $\theta + 2\phi = 45^0$.

7. If $\tan\theta = \frac{1}{\sqrt{3}}$ and $\tan\phi = \frac{1}{\sqrt{15}}$, then $\sin(\theta + \phi)$

$$= \sin 60^0 \cos 36^0.$$

8. Prove that chd $120^0 = \tan 60^0$.

9. If $\tan\theta = \frac{1}{5}$, and $\tan\phi = \frac{1}{239}$, then

$$4\theta - \phi = 45^0.$$

10. Find the numerical values of $\sin 12^0$, $\cos 22^0\frac{1}{2}$, and $\cos 9^0$.

SOLUTION OF, AND FORMULÆ RELATING TO, TRIANGLES.

1. In any triangle

$$\cot A = \frac{c}{a}\operatorname{cosec} B - \cot B.$$

$$\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2} + 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = 1.$$

$$c^2 = (a+b)^2\sin^2\frac{C}{2} + (a-b)^2\cos^2\frac{C}{2}.$$

$$\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

$$\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C.$$

$$\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}.$$

$$\sin(A - B) : \sin C :: a^2 - b^2 : c^2,$$

$$2\frac{a+b}{c}\sin^2\frac{C}{2} = \cos A + \cos B,$$

$$(a+b)\cos C + (a+c)\cos B + (b+c)\cos A = a + b + c.$$

2. If R be the radius of the circle circumscribed about a given triangle ABC, and r the radius of the inscribed circle, then

$$r = 4R \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}.$$

3. In a plane triangle ABC, given the sum of the sides AC, BC; the perpendicular from the vertex C upon the base AB; and the difference of the segments of the base made by the perpendicular; find the sides of the triangle.

4. Given the vertical angle, the perpendicular let fall from the vertical angle on the base, and the rectangle under the segments of the base; find the remaining angles.

5. If r be the radius of the circle inscribed in a triangle, and r_1 r_2 r_3 the radii of three other circles which touch the sides produced, then

$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}.$$

6. In a right-angled triangle, the lines drawn from the acute angles to the point of bisection of the opposite sides are α, β, respectively; find the angles.

7*. In the triangle ABC,

$BC = 236$ feet, angle $ABC = 26°30'$, and angle $ACB = 47°15'$, find the remaining sides.

8. In the triangle ABC,

$AC = 5780$ feet, $AB = 7639$ feet, and angle $ABC = 43° 8'$. Find the remaining side and angles.

9. The angles of a triangle are as the numbers 3, 4, 5, and the radius of the inscribed circle is known. Find the area of the triangle.

10. A triangular field ABC, the sides of which are given, is to be divided into two parts in the ratio of 2 : 1, by a fence passing across from a given point D in AC to BC. Find its length.

11. Find the area of a triangle, having given two angles and a side opposite to one of them.

12. Given the distances from the angles, of the point at which the sides of a plane triangle subtend equal angles; find the sides and the area.

13. The sides of a triangle are in arithmetical progression, and the distance of the centres of the inscribed and circumscribed circles is a mean proportional between the greatest and the least; shew that the sides are in the proportion of $\sqrt{5} - 1 : \sqrt{5} : \sqrt{5} + 1$.

14. In a right-angled triangle, if the hypothenuse be divided into two segments x and y, by the line which bisects the right angle, and $t =$ the tangent of half the difference of the acute angles, then

$$x : y :: 1 + t : 1 - t.$$

15. Given the angles of a triangle, and the radius of the inscribed circle, determine the sides.

16. The triangle ABC has its angles A, B, C in the proportion of $2 : 3 : 4$. Prove that $\cos\frac{A}{2} = \frac{a+c}{2b}$.

17. The angles of a triangle are as the numbers 1, 2, 3; and the perpendicular from the greatest angle on the opposite side is p. Find its area.

18. Determine the triangle, whose sides are three consecutive terms in the series of natural numbers, and whose largest angle is double of the least.

19. If a, b, c be the sides of a triangle, p, q, r perpendiculars from a point within the triangle bisecting the sides, prove that

$$\frac{a}{p} + \frac{b}{q} + \frac{c}{r} = \frac{abc}{4pqr}.$$

20*. If lines be drawn from the angular points of a triangle to the middle points of the opposite sides, the triangle will be divided into six equal parts.

21. If a perpendicular be let fall from the vertex of any triangle on the base, the rectangle under the sides of

the triangle is equal to the rectangle under the perpendicular and the diameter of the circumscribed circle.

22. The hypothenuse of a right-angled triangle is less than the sum of the two sides by the diameter of the inscribed circle.

23. If two triangles have an angle of the one equal to one angle of the other, and also another angle of the one equal to the supplement of another angle of the other, the sides about the two remaining angles shall be proportionals.

24. Given two sides of a triangle and the difference of the angles opposite to them; solve the triangle.

25. Given the area, the perimeter, and an angle; solve the triangle.

26. Given two sides of a triangle and the included angle, find either of the angles into which the given angle is divided by a line drawn from it to the middle point of the opposite side; and adapt the expression to logarithmic computation.

27. The tangents of the angles of a triangle are in harmonical progression; given one of the sides, and the difference of the first and third angle; solve the triangle.

28. The perimeter of a triangle : the diameter of the inscribed circle :: $1 : \tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}$.

29. Given the base, the vertical angle, and the difference of the sides; find the remaining angles.

30. The sides of a triangle are in arithmetical progression, and its area is to that of an equilateral triangle of the same perimeter as $3 : 5$. Find the ratio of the sides and the value of the largest angle.

31. The area of any triangle is to the area of the triangle, the sides of which are respectively equal to the lines joining its angular points with the middle points of the opposite sides, as $4 : 3$.

32. The angles of a plane triangle form a geometrical progression of which the common ratio is $\frac{1}{2}$; express the greatest side in terms of the perimeter.

33. In any triangle

$$\sin nA + \sin nB + \sin nC = \pm 4 \cos \frac{nA}{2} \cos \frac{nB}{2} \cos \frac{nC}{2},$$

if n is of the form $4m + 1$, or $4m + 3$, and

$$= \pm 4 \sin \frac{nA}{2} \sin \frac{nB}{2} \sin \frac{nC}{2},$$

if n is of the form $4m$, or $4m + 2$.

FINDING OF HEIGHTS AND DISTANCES.

1. A river AC, the breadth of which is 200 feet, runs at the foot of a tower CB, which subtends an angle BAC of $25^\circ 10'$ at the edge of the bank. Required the height of the tower.

2. The angles of elevation of a balloon were taken at the same time by three observers, placed respectively at the two extremities and at the middle point of a base measured on the earth's surface. Find an expression for the height of the balloon.

3. In order to ascertain the height of a mountain, a base was measured of 2761 feet, and at either extremity of this base were taken the angles formed by the summit and the other extremity; these were $58^\circ 29'$ and $111^\circ 52'$; also at the extremity from which the latter angle was taken, the angular height of the mountain was $11^\circ 18'$. Required the mountain's height.

4. A person standing at the edge of a river observes that the top of a tower on the edge of the opposite side subtends an angle of 55° with a line drawn from his eye parallel to the horizon; receding 30 feet, he finds it to subtend an angle of 48°. Determine the breadth of the river.

5. A person on a tower can see the top of a pillar of known altitude, from which he wishes to know the distance, and the height of the tower. He can see also an object on the horizontal plane from which he has formerly observed the angular distance of the top of the pillar from that of the tower. Shew how he may find the required distances, having with him an instrument for measuring angles.

6. A person on the top of a tower, the height of which is 50 feet, observes the angles of depression of two objects on the horizontal plane, which are in the same straight line with the tower, to be 30^0 and 45^0 respectively. Find their distances from each other and from the observer.

7. Three objects A, B, C, form an isosceles triangle whose vertex is B, and whose angles are as the numbers 4, 1, 1; an observer walking from A toward C, measures a base AD of a feet, and observes the angle BDC; he then advances to E, b feet further, and observes that the angle BEC = the supplement of BDC. From these observations find the sides of the triangle.

8. A person walking from C to D on the horizontal road can plainly see the summit of a hill A from every point except E, where he can *just* see it over a hill B. He measures EC, and at C observes the angles of elevation of B and A, as well as the angles ACB, ACE. At E he observes the angle AEC. Shew how to find the heights of the hills.

9*. In ascending a tower of known height, a person observes from a window the angle of depression of a point in the horizontal plane upon which the tower stands; when he arrives at the top of the tower he observes the angle of depression of the same point; shew how to find the height of the window above the ground.

10. A person wishing to ascertain the height of a tower standing on a declivity, ascends 80 feet from its base, and it then subtends an angle of 30^0. The inclination of the side of the hill to the horizon being 15^0, find the height of the tower.

11. The elevation of a steeple standing on a horizontal plane is observed, and at a station a feet nearer to it its elevation is found to be the complement of the former. On advancing b feet nearer still, the elevation is found to be double the first; shew that the height of the steeple is

$$\left\{(a+b)^2 - \frac{a^2}{4}\right\}^{\frac{1}{2}}.$$

12. From the summit of a tower, the height of which is 108 feet, the angles of depression of the top and bottom of a column, standing on the same horizontal plane with the tower, are observed to be 30^0 and 60^0 respectively. Required the height of the column.

13. The top of a tower is visible from three stations A, B, C, in the same horizontal plane; at each of the stations the angular distance of the top of the tower from each of the other two stations is observed; given the distance between A and B, and the height of the tower, it is required to find the distance of C from each of the other stations and from the tower.

14. A person travelling along a straight road observes the elevation of a tower, the nearest distance of which from the road is known. At the same time he also observes the angular distance of the top of the tower from an object in the road. Required the height of the tower.

15. Describe the observations and calculations, necessary for determining the breadth of a river, from stations upon one of its banks.

16. From a station B, at the base of a mountain, its summit A is seen at an elevation of 60^0; after walking 1 mile towards the summit, up a plane making an angle of 30^0 with the horizon, to another station C, the angle BCA is observed to be 135^0. Find the height of the mountain in yards.

17. An object 6 feet high, placed on the top of a tower, subtends an angle the tangent of which is ·015, at a place the horizontal distance of which from the foot of the tower is 100 feet; determine the height of the tower.

ILLUSTRATIONS OF THE CIRCULAR MEASURE.

1. Find the number of degrees, minutes and seconds, in the angles, the circular measures of which are $\frac{\pi}{7}$, 1.5, 2, $\pi + 1$, and 3.14 respectively.

2. Reduce to the circular measure the following angles; 14°, $15^\circ\ 30'$, 120°, $17^\circ\ 8'$, and $92^\circ\ 3'$.

3. If two-thirds of a right angle be assumed as the angular unit, what will be the numerical value of an angle of 45°?

4. Determine the angular unit by the assumption of which the following equation would be numerically true,

$$\text{angle} = 2\,\frac{\text{subtended arc}}{\text{radius}}.$$

5. Find the circular measure of $1'$.

6. It may be shewn, that if a very small angle θ is expressed according to the circular measure $\sin\theta = \theta - \frac{\theta^3}{6}$ nearly; what change must be made in the formula if θ is expressed in seconds?

MISCELLANEOUS PROBLEMS.

1. Given three lines drawn from any point within a square to three of its angular points; determine a side of the square.

2. Express the diagonals of a quadrilateral inscribed in a circle in terms of the sides.

3. The excess of the sine above the versed sine, in angles less than 90°, is greatest when the angle $= 45^\circ$. Prove this and find the value of the maximum excess.

4. In any quadrilateral figure, the square of one side is less than the sum of the squares of the other sides, by

twice the sum of the products of these sides taken two together and multiplied by the cosine of the angle between them.

5. Find the angle at which a side of a pyramid is inclined to the base, the sides being equilateral triangles, and the base a square.

6. A vessel observed another α^0 from the North, sailing in a direction parallel to its own. In p hours its bearing was β^0, and in q hours afterwards γ^0 from the North. To what point of the compass were the vessels sailing?

7. Three circles, whose diameters are $\sqrt{3}-1$, $\sqrt{3}+1$, and $3-\sqrt{3}$ respectively, touch each other in the points A, B, C. Find the area of the triangle ABC.

8. Two regular polygons of the same number of sides being described, the one within, and the other without, the same circle; what will be the number of sides when the space included between the two polygonal boundaries is $\frac{1}{3}$ of the interior polygon?

9. Coasting along shore, observed two head-lands: the first bore N.N.W., and the second N.E. by E.; then steering 12 miles in the direction E.N.E., the first bore N.W., and the second N.E. Shew how the bearing and distance of the two head-lands from each other may be found.

10. Compare the areas of decagons inscribed in, and circumscribed about, a circle.

11. A semicircle is divided into three arcs A, B, C, whose cosecants are in harmonical progression, shew that

$$\sin\frac{A}{C} : \sin\frac{C}{2} :: \sin\frac{A-B}{2} : \sin\frac{B-C}{2}.$$

12. A regular polygon is described in a circle, and the tangent of half the acute angle which a side subtends at the circumference $= t$; shew that a side of the figure : the diameter of the circle :: $2t : 1+t^2$.

13. Required the perpendicular from the vertex upon the base of a triangular pyramid, all the sides of which are equilateral triangles of given area.

14. A circle is inscribed in an equilateral triangle, an equilateral triangle in the circle, a circle in the latter triangle, and so on *ad infinitum;* if r, r_1, r_2, r_3...be the radii of the circles, prove that

$$r = r_1 + r_2 + r_3 + \ldots\ldots\ldots$$

15. If a, b, c, d be the four sides of a quadrilateral figure inscribed in a circle, and $2s = a + b + c + d$, and A be the angle contained by a and d, then

$$\tan \frac{A}{2} = \left\{\frac{(S-a)(S-d)}{(S-b)(S-c)}\right\}^{\frac{1}{2}},$$

$$\text{and the area} = \sqrt{(S-a)(S-b)(S-c)(S-d)}.$$

16. The sides of a plane triangle are as 3, 5, 6; compare the radii of the inscribed and circumscribed circles.

17. If from any point O within a triangle, three straight lines be drawn from the angles A, B, C, meeting the opposite sides in a, b, c, then will

$$\frac{Oa}{Aa} + \frac{Ob}{Bb} + \frac{Oc}{Cc} = 1.$$

18. In any right-angled plane triangle, twice the side of the inscribed square is an harmonical mean between the sides containing the right angle.

19. Prove that the area of a regular polygon *inscribed* in a circle is a geometrical mean between the areas of an inscribed and of a circumscribed polygon of half the number of sides; and that the area of a regular polygon *circumscribed* about a circle, is an harmonical mean between the areas of an inscribed one of the same number of sides, and of a circumscribed one of half that number.

20. If the side of a pentagon inscribed in a circle be 1, the radius is

$$\sqrt{\frac{5+\sqrt{5}}{10}}.$$

21. Given the radius of the circumscribed circle, and the three angles of a triangle; find expressions for the three sides.

22. If R, r be the radii of the circumscribed and inscribed circles of a regular polygon of n sides, and R', r' the corresponding radii for a regular polygon of $2n$ sides, and of the same perimeter as the former, then

$$Rr' = R'^2, \text{ and } R + r = 2r'.$$

23. An indefinite area can be divided into no other regular figures than triangles, squares, and hexagons.

24. If A, B, C be the angular points of a triangle, a, b, c points in the sides respectively opposite to them, prove that the lines Aa, Bb, Cc will intersect in a point, if

$$\frac{Ac}{Ab} \cdot \frac{Ba}{Bc} \cdot \frac{Cb}{Ca} = 1.$$

25. If R, r be the radii of circles circumscribed about and inscribed in the same plane triangle, prove that the distance between the centres of the circles $= \sqrt{R^2 - 2Rr}$.

26. Adapt to logarithmic computation the expression

$$\frac{\sqrt{a-b}}{a+b} + \frac{\sqrt{a+b}}{a-b}.$$

27. In the equation

$$nx = (\sqrt{1+x} - 1)(\sqrt{1-x} + 1),$$

prove that if $n = \tan \dfrac{a}{2}$, $x = \sin 2a$.

28. If a quadrilateral is capable of having a circle inscribed in it, the sums of the opposite sides are equal to one another; and if, besides, it is capable of having one circumscribed about it, its area equals the square root of the continued product of the sides.

29*. If $\dfrac{\sin(a-\beta)}{\sin\beta} = \dfrac{\sin(a+\theta)}{\sin\theta}$, shew that

$$\cot\beta - \cot\theta = \cot(a+\theta) + \cot(a-\beta).$$

30. Shew that

$$\left(\tan\frac{\theta}{3}+\tan\frac{2\pi+\theta}{3}+\tan\frac{4\pi+\theta}{3}\right)\left(\cot\frac{\theta}{3}+\cot\frac{2\pi+\theta}{3}+\cot\frac{4\pi+\theta}{3}\right)=9.$$

31. A lamp on the top of a pole 32 feet high is just seen by a man six feet in height, at a distance of 10 miles; find the earth's radius.

32. A ship, the height to the summit of the top mast of which from the water is 90 feet, is sailing directly towards an observer at the rate of 10 miles an hour; from the time of its first appearance in the offing till its arrival at the station of the observer is 1 hour 12 minutes: find approximately the earth's radius.

STATICS.

COMPOSITION AND RESOLUTION OF FORCES.

1. THREE forces acting in the same plane keep a point at rest; the angles between the directions of the forces are 135^0, 120^0, and 105^0; compare their magnitudes.

2. Two forces sustain each other by means of a string passing over a tack; prove that either force : pressure on tack :: $\frac{1}{2}$: cosine of half the angle between the directions of the forces.

3. A body is suspended from a given point in a horizontal plane, by a string of known length, which is thrust out of its vertical position by a rod (without weight) acting from a given point in the plane against the body; find the tension of the string.

4. If θ be the angular distance of a body from the lowest point of a circular arc in a vertical plane, the force of gravity in the direction of the arc : that in the direction of the chord :: $2 \cos \frac{\theta}{2} : 1$.

5. AB is a given horizontal line, BC a rod without weight moving freely in a vertical plane about B. A weight is suspended by a string fixed at A and passing over the end C of the rod. Find the position of equilibrium.

6. A string PAQ is knotted to a fixed point A, and drawn in different directions by the forces P and Q, in such a manner that the pressure on $A = \frac{P+Q}{2}$; find the angle PAQ.

7. Two forces are in the ratio of 3 : 2; find the angle between their directions, when the resultant is a mean proportional between them.

8. Ten men of equal strength wishing to pull down a tree of given height, and at the same time to avoid all danger from its fall, fix two ropes at its top, one of which reaches the ground making an angle of 60° with the tree, and the other 45°. If four men pull at the former towards the south, and six at the latter towards the south-east, towards what point of the compass will the tree fall?

9. Given the resultant of two forces, their sum, and the angle between their directions; find the forces.

10. Three forces acting upon a point, keep it at rest; and they are in the ratios of $\sqrt{3}+1 : \sqrt{6} : 2$. Find the angles at which they are respectively inclined to each other.

11. A small ring is attached to one end of a string of given length, the other end of which is fixed to a given point. Another string is fixed to a given point in the same horizontal line as the former, and this string passing through the ring supports a weight; find the position of the ring.

12. Four forces represented by 1, 2, 3, and 4, act in the same plane on a point. The directions of the first and third are at right angles to each other; and so are the directions of the second and fourth; and the second is inclined at an angle of 60° to the first. Find the magnitude and direction of the resultant.

13. A circular hoop is supported in a horizontal position, and three weights of 4, 5, and 6 pounds respectively are suspended over its circumference by three strings knotted together at the centre of the hoop. Find the angles between the strings when there is equilibrium.

14. Two equal weights are supported by a string which passes over three tacks, forming a vertical isosceles triangle of which the base is horizontal and the angle opposite to the base 120°. Find the pressure on each of the tacks.

15. Three forces represented by the numbers 3, 5, 9 cannot under any circumstances produce equilibrium upon a point.

16. If three forces in equilibrium upon a point are represented by the numbers 3, 4, 5, respectively, two of them are perpendicular to each other.

17. The resultant and sum of two forces being given, and also the angle which one of them makes with the resultant; it is required to determine the forces and the angle at which they act.

PRINCIPLE OF THE LEVER.

1. AC, CB are the equal arms of a straight lever whose fulcrum is C: to C a heavy arm CD is fixed perpendicular to AB. Prove that if a weight be suspended to the extremity A, and the system be in equilibrium, the tangent of the inclination of CD to the vertical will be proportional to the weight.

2. Four weights, 1, 3, 7, 5 are placed at equal distances on a straight lever. Determine the fulcrum.

3. There are n weights, W_1, $W_2 \ldots\ldots W_n$ in geometrical progression, and W_1 placed at A, one extremity of a lever, balances W_2 placed at B, the other extremity. Prove that a weight equal to the first $n-1$ weights, if placed at A, will balance a weight equal to the last $n-1$, if placed at B.

4. The lever AC (without weight) turning about the fulcrum C, has two given weights W, W' suspended from the extremity A and the middle point B respectively, and is kept at rest by the given weight P acting at A by means of a string passing over a tack at D: CD is horizontal and equal to AC: find the position of equilibrium.

5. A uniform bent lever, when supported at the angle, rests with the shorter arm horizontal. If the shorter arm were twice as long, it would rest with the other horizontal. Compare the lengths of the arms, and find the angle between them.

6. A straight lever of uniform thickness, the length and weight of which are given, has two weights P and Q attached to its extremities, and is sustained partly by a fulcrum at a given point, and partly by another fulcrum, on which it presses with a given force; required the position of this latter fulcrum.

7. A uniform bent lever ABC, containing a given angle at B, and having its arms of given weights and lengths, hangs freely by the extremity A: find the position of equilibrium.

8. A straight lever is sustained on a fulcrum at the middle point, and is kept at rest by two given weights; where must they be placed, in order that the distance of the one from the fulcrum may equal the distance of the other from the extremity? And where must the fulcrum be placed, if the position of the weights be reversed?

9. A uniform rod of given weight and length suspended at a given point, is drawn out of the vertical position by a given force, acting upon its lower extremity by means of a string, which passes over a peg at a given point in the same horizontal line with the axis of suspension; find the angle through which the rod is drawn.

10. Two weights, 3 and 4, balance on the extremities of a lever 4 feet long; find the fulcrum.

11. A uniform beam of given weight and length is moveable about its middle point; a given weight is hung by a string to one end; find to what extent the other end of the beam must be lengthened in order that there may be equilibrium.

12. In question 8, page 66, determine the moment of the force which is effective in pulling down the tree.

13. A bar weighs a oz. per inch. Find its length when a given weight na, suspended at one end, keeps it in equilibrium about a fulcrum at a distance of b inches from the other end.

14. A beam, 30 feet long, balances about a point at one third of its length from the thicker end; but when a weight

of 10 lbs. is suspended from the smaller end, the fulcrum must be moved 2 feet towards it in order to maintain equilibrium. Find the weight of the beam.

15. At what point of a tree must a rope of given length be fixed, so that a man endeavouring to pull down the tree may have the greatest advantage?

16. A uniform beam 18 feet long, rests in equilibrium upon a fulcrum 2 feet from one end, having a weight of 5 lbs. at the end furthest from the fulcrum and one of 110 lbs. at the other. Find the weight of the beam.

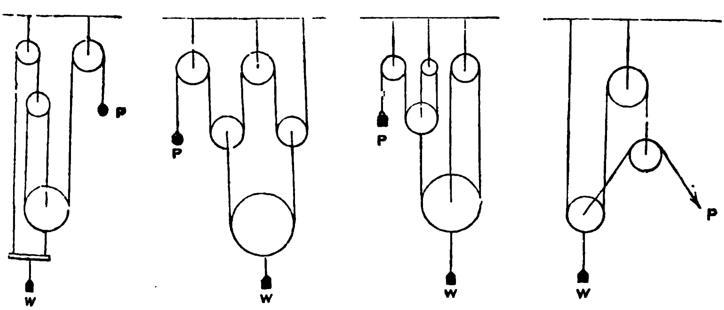
17*. Three equal rods without weight are attached together, each by one extremity to the extremities of the other two, so as to form a rigid framework; the rods make equal angles with each other, and the framework is suspended in a vertical plane by a pivot at its centre; if three weights be suspended from the outer extremities of the three rods find the position of equilibrium.

18. One end of a beam is connected with a fixed point by a hinge, about which the beam can revolve in a vertical plane; the other end is attached to a weight, by means of a string passing over a tack in the same vertical plane; find the position of equilibrium.

MECHANICAL POWERS.

1. What force must be exerted to sustain a ton weight on a screw, the thread of which makes 158 turns in the course of 12 inches, and which is acted on by an arm 6 feet long?

2. In the annexed systems of pullies, compare *P* and *W*.



3. A weight W is sustained upon an inclined plane by a force P, acting by means of a wheel and axle, placed at the top, in such a manner that the string attached to the weight is parallel to the plane. Given R and r the radii of the wheel and axle, find the inclination of the plane.

4. What force is necessary to support a weight of 60 lbs. on a plane inclined at an angle of 60° to the horizon, the force acting horizontally?

5. Find the angle at which a given force must act, in order that it may just support a given weight on a given inclined plane.

6. When a given weight is sustained on a given inclined plane by a force in a given direction, find the pressure on the plane.

7. Given the weight, and the magnitude and direction of the sustaining force; find the angle of the plane.

8. Find the ratio of P to W in the various systems of pullies, taking into account the weights of the pullies.

9. Find the ratio of P to W in the single moveable pully, the strings making a given angle α with the horizon.

10. In the second system of pullies, if there be ten strings to the block, what power will support a weight of 1000 lbs.?

11. In the third system of pullies, there being 6 pullies, what weight can be supported by a weight of 12 lbs.?

12. What must be the length of a lever at the extremity of which a force of 1 lb. will support a weight of 1000 lbs. on a screw; the distance between two contiguous threads being $\frac{3}{4}$ inch?

13. Two weights sustain each other on two inclined planes having a common altitude, by means of a string parallel to the planes; compare the pressures on the planes.

14. In the third system of pullies, there being 8 equal heavy pullies, find the ratio of the weight of one of the

pullies to the weight supported, in order that the latter may be supported by the weight of the pullies alone.

15. What weight is that, which it would require the same exertion to lift as to draw a weight of 4 lbs. up a plane inclined at an angle of 30^0 to the horizon?

16. On an inclined plane the pressure, force, and weight are as the numbers 4, 5, 7; find the inclination of the plane to the horizon, and the inclination of the force's direction to the plane.

17. Find the inclination to the horizon of the thread of a screw, which with a force of 5 lbs. acting at an arm of 2 feet can support a weight of 300 lbs. on a cylinder of 3 inches radius?

FRICTION.

1. If the inclination to the horizon of a plane on which a body is placed, is slowly increased till the body begins to move, its tangent, at the instant of the commencement of motion, is equal to the coefficient of friction.

2. A given force (P) acting parallel to the horizon, just sustains a body of given weight (W) on a rough inclined plane, the angle of which is θ. The same body will just rest without support on a plane, of the same material, the inclination of which is a. Determine θ.

3. A heavy body is to be conveyed to the top of a rough inclined plane, the angle of inclination being a; prove that if the coefficient of friction be greater than $\dfrac{\sin\left(45^0 - \dfrac{a}{2}\right)}{\sin\left(45^0 + \dfrac{a}{2}\right)}$, it will be easier to lift the body than to drag it up by means of a cord parallel to the plane.

CENTRE OF GRAVITY.

1. Out of a square it is required to cut a triangle having one side for its base, such that the centre of gravity of the remainder shall be in its vertex.

2. The sum of the products of the weights of any number of particles by the squares of their respective distances from a given point, is less when that point is the centre of gravity of the system than when it is any other.

3. The locus of the centres of gravity of all right-angled triangles on the same hypothenuse (a) is a circle of radius $\frac{a}{6}$.

4. Determine the length of a straight line drawn through the centre of gravity of a given isosceles triangle, making a given angle with the base, and terminated by the sides.

5. Three weights of 1, 2, and 3 lbs. are placed at the corners of an equilateral triangle; determine the distance of their centre of gravity from each of the corners of the triangle in terms of the length of the side.

6. Two weights of 5 and 7 lbs. are connected by a rod without weight measuring 6 feet, find their centre of gravity, and determine the change which will take place in its position if 1 lb. be added to the smaller weight.

7. If the sides of a triangle ABC be bisected in the points D, E, F; the centre of the circle inscribed in the triangle DEF will be the centre of gravity of the perimeter of the triangle ABC.

8. CA and CB are the arms of a uniform bent lever; determine the distance of the centre of gravity from C, in terms of the lengths of the arms and the angle ACB.

9. If G be the centre of gravity of the triangle ABC, then

$$3(GA^2 + GB^2 + GC^2) = AB^2 + AC^2 + BC^2.$$

10. If the sides of a triangle are 3, 4, and 5; determine the distance of its centre of gravity from either of the angles.

11. If three equal bodies are placed in the corners of a triangle, their centre of gravity coincides with that of the triangle.

MISCELLANEOUS PROBLEMS.

1. A uniform beam AB, of given weight, is moveable in a vertical plane round a hinge at A, and is kept in a given position by a weight P, acting by means of a string attached to the beam at B, and passing over a pully at C, a point in the same vertical line as A. Find P.

2. A solid cone of given weight and having a base of given radius, stands upon a plane inclined at an angle of 30^0 to the horizon and is prevented from sliding; determine its height so that it may just not fall over.

3. A string having its extremities fastened to the ends of a uniform bar of known weight, passes over four tacks so as to form with the bar a regular hexagon, the bar being horizontal. Find the tension of the string, and the vertical pressure on each tack.

4. Two weights P and Q balance each other upon the surface of a sphere by a string of given length, passing over the highest point. Required the position of equilibrium.

5. Two weights sustain each other upon two inclined planes having a common altitude, by means of a string which is attached to each; find their position, taking into account the weight of the string, which is supposed to be uniform.

6. AD is horizontal, DC vertical, Q a weight connected with one extremity of a beam AB (moveable in a vertical plane about the point A,) by a string passing over a pully at C in such a manner that CB is vertical. Find the relation between Q and the weight of the beam.

7. A weight W is sustained on an inclined plane by three forces, each equal to $\frac{P}{3}$, one acting vertically upwards, another parallel to the plane, and the third parallel to the horizon; required the plane's inclination.

8. If three parallel forces acting at the angular points A, B, C of a triangle be respectively proportional to the opposite sides a, b, c; prove that the resultant may be supposed to act at a point, the distances of which from the points A, B, C are respectively

$$\frac{2bc}{a+b+c}\cos\frac{A}{2},\ \frac{2ca}{a+b+c}\cos\frac{B}{2},\ \text{and}\ \frac{2ab}{a+b+c}\cos\frac{C}{2}.$$

9. Two uniform beams, of given weights and lengths, have their upper extremities in contact with two smooth parallel vertical planes; and their lower extremities act upon each other. Required the distance between the planes when the beams rest at right angles to each other.

10. A uniform rod of given weight and length, has a weight attached to a certain point of it, and is placed with one end against a smooth vertical wall, the other upon a smooth horizontal plane; find the position of the weight, when a given horizontal force is just sufficient to prevent the rod from sliding when in a given position.

11. A uniform beam AB, of given length and weight, rests with one end on a given inclined plane, and the other attached to a string AFP passing over a pully at F given in position. Knowing the weight P fixed to the other end of the string, find the position in which the beam rests.

12. AC and BD are two beams, of given weights, moveable in a vertical plane about the fixed points A and B in the same horizontal line; BD rests upon AC as a prop. Find the position of equilibrium.

DYNAMICS.

ELEMENTARY PRINCIPLES.

1. A BODY passes uniformly over a distance of 200 yards in the time $1^h. 6^m$: what is the numerical value of its velocity, according to the usual conventions respecting the units of space and time?

2. A body is observed to describe uniformly a feet in n seconds; supposing the unit of time to be 1 minute, what must be the unit of distance in order that the numerical value of the body's velocity may be 1?

3. A man walks with a velocity represented by 2, and he finds that he walks 7 miles in 2 hours; if 1 foot be the unit of length, what is the unit of time?

4. A particle is moving with such a velocity, and in such a direction, that the resolved parts of its velocity in the directions of two lines perpendicular to each other are respectively 3 and 4; determine the direction and velocity of the particle's motion.

5. If 1 yard be taken as the unit of length, and 1 minute as the unit of time, what will be the numerical value of the accelerating force of the earth's attraction?

6. A body is proceeding uniformly along a certain straight line; suddenly it is observed to move off with unaltered velocity in a direction making an angle of 60^0 with the former direction of its motion; determine the direction and the accelerating force of the impulse to which this change of the motion is due.

7. If 1000 oz. be taken as the unit of weight, what will be the weight of the body the mass of which is numerically equal to unity?

8. A is a more powerful and a heavier man than B; the greatest weights which they can lift are as 8 : 7, and their own weights are as 7 : 6. Which is likely to be the faster runner of the two?

FALLING BODIES.

1. Find the depth of a well by dropping a stone into it, the velocity of sound being supposed to be known.

2. A body is projected upwards with a velocity which will carry it to a height $2g$ feet; after how long a time will it be descending with a velocity g?

3. Find the velocity with which a body must be projected upwards from the foot of a tower, so as to meet halfway another body let fall at the same time from the top of the tower.

4. A balloon is ascending vertically with a given velocity, and a body is let fall from it, which reaches the ground in t'': find the height of the balloon at the moment of the body leaving it.

5. A body is observed to fall the last a feet of its descent from rest in t'': find the height from which it fell.

6. A body has fallen through a distance of half a mile; what was the space described in the last second?

7. The space described in the fifth second of its fall was to the space described in the last second but four as 1 : 6; what was the whole space described by the body?

8. A body is projected upwards with a velocity of 64 feet in $1''$; how far will it ascend before it begins to return?

9. With what velocity must a stone be projected from the top of a tower, 250 yards above the sea, that it may reach the water in $6''$?

10. A body falls through a distance a feet at two different places on the earth's surface; and it is observed that the time of falling is t'' less, and the velocity acquired m feet greater at one place than at the other: compare the force of gravity at the two places.

11. A stone dropped from a bridge strikes the water in $2\frac{1}{2}''$; what is the height of the bridge? Also, if the stone be projected downwards with a velocity of 3 feet per second, in what time will it strike the water?

12. Two balls are projected at the same instant towards each other, from the two extremities of a vertical line, each with the velocity which would be acquired in falling down it. Where will they meet?

13. A falling body is observed to describe in the n^{th} second of its fall a space equal to p times that described in the $\overline{n-1}|^{th}$: required the whole space described.

14. A body is projected vertically upwards, and the time between its leaving a given point and returning to it again is given; find the velocity of projection and the whole time of motion.

MOTION ON AN INCLINED PLANE.

1. Determine the velocity with which a body must be projected down an inclined plane, that it may describe the plane in the time in which the altitude would be described by a body falling from rest.

2. The length of an inclined plane is 400ft., its height 250. A body falls from rest from the top of the plane; what space will it have fallen through in $3''.5$? and what velocity will it have acquired, when it is within 50 feet of the bottom of the plane?

3. In an inverted parabola the time of descending down any chord from a point in the curve to the vertex, is equal to the time of falling freely to a horizontal line

which is at a distance below the vertex equal to the latus rectum.

4. A body being projected down an inclined plane with the velocity which would be acquired in falling down its perpendicular height, the time of descent is found to be that of falling down the height. Required the plane's inclination.

5. Determine that diameter of a vertical circle, down the latter half of which a body falls in the same time as down the whole vertical diameter.

6. Two bodies A and B descend from the same extremity of the vertical diameter of a circle, one down the diameter, the other down the chord of 30°. Find the ratio of A to B when their centre of gravity moves down the chord of 120°.

7. Determine that point in the hypothenuse of a right-angled triangle, having its base parallel to the horizon, from which the time of a body's descent down an inclined plane to the right angle is least.

FALLING BODIES CONNECTED BY A STRING.

1. Twelve pounds weight is so distributed at the extremities of a string passing over a pully, that the more loaded end descends through 7 feet in as many hours. What weight is at each end of the string?

2. With what weight must a given weight W be connected by a string passing over a single fixed pully, so as to describe the same space in a given time as when it descends freely down a given inclined plane?

3. A weight of 10lbs. is attached to one end of a string; find the weight which must be attached to the other, in order that when the system is suspended from a fixed pully, the accelerating force may be half that of gravity.

4. Two weights connected by a string passing over a pully are in motion, and it is observed that the space through

which the heavier descends in the n^{th} second from the commencement of the motion is a feet; compare the weights.

5. A bucket is raised from a well of unknown depth by means of a weight (W) connected with it by a string and descending over a pully, and the bucket is raised in T seconds; when a weight W' is substituted for W, the time of raising the bucket is found to be T' seconds; from these data calculate the weight of the bucket and the depth of the well.

6. A given inclined plane has a pully at its highest point, over which a string passes connecting two weights, one of which rests on the plane and the other hangs freely; having observed the velocity with which the system moves, determine what would be the velocity if the two weights were made to change places.

7. P draws Q up a given inclined plane by means of a string and pully, P falling vertically; where is Q when its position is such that the string being cut it will just reach the top of the plane?

PROJECTILES.

1. A number of balls of given weight are projected at the same instant in given directions with given velocities; find the height of their centre of gravity at a given time.

2. A convoy moving uniformly along a road which runs east and west, is perceived at the instant it is due south of a battery; at what elevation, and towards what point of the compass, must a cannon be fired at the same instant, so as just to hit it? (The velocity with which the ball issues from the cannon's mouth is supposed to be known.)

3. A body projected in an oblique direction along a smooth inclined plane, will trace out a parabola upon the plane.

4. Determine the angle of projection for which the horizontal range will be greatest, the velocity of projection being given.

5. If the velocity of projection be given, determine the angle so that the focus of the parabolic path may be in the horizontal plane through the point of projection.

6. A body is projected at an angle of 30^0 to the horizon, and with the velocity which would be acquired in falling through a vertical height of 10 feet, determine the latus rectum of the parabola described.

7. The latus rectum of the path of a projectile is a, and the horizontal range b, determine the velocity and direction of projection.

8. Determine the angle of projection so that the horizontal range and the latus rectum may be equal.

9. A ball fired from a gun just clears a wall of known height, and on picking up the ball it is found that the distance from the wall of the point at which it fell is exactly equal to the distance from the wall of the person who fired the gun; supposing this distance ascertained, determine the angle of elevation of the gun.

10. A man fires a ball into the air, wishing it to fall at a certain point; he finds however that it falls only half way; he accordingly elevates his gun through an angle of 15^0, and the ball now falls exactly at the point desired. Find the original elevation of the gun.

11. One boy (A) throws a ball to another (B). B endeavours to throw it back, and projects it at exactly the same elevation as A, but the ball only reaches $\frac{7}{8}$ths of the required distance; compare the strengths of A and B.

12. A projectile is observed to strike the ground at an angle of 45^0, and is ascertained to be moving at the instant of impact with a velocity of 400 feet per second; determine the position of the focus and the latus rectum of the parabolic path.

13. A body is projected from a point in a plane inclined at a given angle to the horizon; find the angle of projection in order that the focus of the parabolic path may be in the inclined plane.

14. A body is projected in a given direction, and with a given velocity, from a point in a plane making a given angle with the horizon; determine the distance of the body from the plane at any time (t), and the time which elapses before the body strikes the plane.

15. A body is projected from a point in an inclined plane, with a velocity V and at an angle α with the horizon; prove that, if β be the angle of the plane, the range on the inclined plane $= \dfrac{V^2}{g\cos^2\beta}\{\sin(2\alpha-\beta)-\sin\beta\}$; and hence shew that the range is greatest when $\alpha = 45^0 + \dfrac{\beta}{2}$.

16. A body is projected with a given velocity along an inclined plane of given length; find the latus rectum of the parabola described by the body after leaving the plane.

17. Find the velocity and direction of projection of a ball, that it may be 100 feet above the earth at one mile distance, and may strike the ground at a distance of three miles.

IMPACT.

1. The centres of two elastic balls M and M' move along the same straight line with velocities V and V' respectively. Find the velocity of each after impact, when

$$6M = 5M',\quad V = 7 \text{ feet per second},\quad 4V + 5V' = 0, \text{ and}$$
$$\text{modulus of elasticity} = \tfrac{2}{3}.$$

2. Two equilateral triangles are placed in the same vertical plane, and with their bases at a given distance from each other upon the same horizontal line: an inelastic body falls down the side of the first, moves along the space between the bases and up the side of the second triangle, the vertex of which it just reaches; given the side of the first triangle, find that of the second, and likewise the whole time of motion.

3. A perfectly elastic ball is let fall from a given point in the directrix of a parabola, the axis of which is vertical, and is reflected at the curve; determine the latus rectum of the parabola described.

4. The position of a ball on a triangular billiard table being given; it is required to shew that there are three directions, in any one of which if the ball be struck, it will pursue the same course after being twice reflected at each side. The ball to be considered perfectly elastic.

5. *PQ* is a vertical line terminating in a hard horizontal plane at *Q*; a perfectly elastic ball being dropped from *P* meets another perfectly elastic ball rebounding with a known velocity from *Q*, and both are reflected back; find where they must meet in order that they may thus rebound from one another continually.

6. *A* and *B* are two balls of given elasticity; what must be the magnitude of a third ball, in order that the velocity communicated from *A* to *B* by the intervention of this ball may be equal to that communicated immediately from *A* to *B*? Determine also the limits within which the problem is possible.

7. Two balls are projected at the same instant from two given points in a horizontal plane, and in opposite directions, so as to describe the same parabola. What must be their relative magnitude, and their elasticity, in order that one of them may return through the same path as before, and the other descend vertically after impact?

8. A perfectly elastic body is projected from a point in a plane inclined at an angle α to the horizon; determine the angle at which it must be projected so that after striking the plane it may be reflected vertically upwards.

9. If the modulus of elasticity be $\frac{1}{3}$, at what angle must a body be incident on a hard plane, that the angle between the directions before and after impact may be a right angle?

10. An inelastic body is projected from one angle along the side of a hexagon; and it moves in the interior of the hexagon, describing the different sides in succession; prove that the time of describing the first side : time of describing the last :: 1 : 32.

11. An imperfectly elastic body descending vertically from rest, meets a horizontal plane, which is moving uniformly in an opposite direction; given the distance between the body and the plane at first, and the modulus of elasticity, find the velocity of the plane, so that the body may return to the point from which it fell.

12. A number of elastic balls are placed in a right line. The first is made to start with a given velocity; determine the ratio of the balls, so that its momentum may be equally divided among the remainder.

13. If an elastic ball be projected at an angle θ and with velocity V; prove that the sum of all the horizontal ranges

$$= \frac{V^2 \sin 2\theta}{g(1-e)}.$$

14. Two elastic balls A and B, (such that $A = 3B$) are placed on a horizontal table. A impinging on B at rest drives it perpendicularly against a hard vertical plane, and it meets A in returning at half its original distance. Find the modulus of elasticity.

15. P and Q are two weights connected by a string passing over a fixed pully, whereof P is the greater; at the end of t'' an additional weight (q) is suddenly affixed to Q. Find the velocity of P at any assigned time.

16. A ball (elasticity e) is projected from a given point in the circumference of a circle: after being reflected twice at the circumference it returns to the point of projection. Required the direction of projection.

17. Prove that if a body be projected from one extremity of the diameter of a circle, in a direction making an angle θ with the diameter such that the body after one reflexion at the curve passes through the other extremity, then

$$\sin\theta = \sqrt{\frac{e}{1+e}}: \text{ (}e\text{ the modulus of elasticity.)}$$

18. If two perfectly elastic balls, the masses of which are in the ratio of 1 : 3, meet directly with equal velocities, the larger one will remain at rest.

MISCELLANEOUS PROBLEMS.

1. A body, projected in the direction of a uniform force, describes P and Q feet in the p^{th} and q^{th} seconds respectively. Find the magnitude of the force and the velocity of projection.

2. Find the velocity acquired by an inelastic body descending through a system of three planes, the first being vertical, the second inclined at 45°, and the third at 15° to the horizon.

3. Find the elasticity of two bodies A and B, and their proportion to each other, so that when A impinges upon B at rest, A may remain at rest after impact, and B move on with an n^{th} part of A's velocity.

4. Two weights are connected by a string which passes through a hole in a horizontal plane, one rests upon the plane, the other falls under the action of gravity; determine the motion.

5. Uniform force is defined as that which generates equal velocities in equal *times;* would it be correct to define it as that which generates equal velocities while the body moves through equal *spaces?*

6. A rocket ascending vertically, with an initial velocity of 100 feet per second, explodes when at its greatest height; the interval between the sound of the explosion reaching the place of starting and a place a quarter of a mile distant is 1 second. Determine the velocity of sound.

7. An inelastic body moving along the interior of a regular polygon, will describe all the sides uniformly, if it commences moving uniformly; but the velocities with which it describes the successive sides will decrease according to a geometrical progression. Prove this, and in the case of a hexagon find the ratio of the velocities with which the first and last sides are described.

We subjoin here a few illustrations of the formula for the time of oscillation of a pendulum. If the length of the pendulum be l, and the bob be made to describe a cycloidal arc, or the arc of vibration be so small that the difference between it and a cycloidal arc may be neglected, the formula for the time of a semi-vibration is

$$T = \pi\sqrt{\frac{l}{g}}.$$

It will be assumed, that above the earth's surface the force of gravity varies inversely as the square of the distance from the earth's centre; so that if g' be the value of gravity at a small height h above the earth's surface, and R the earth's radius,

$$g' = g\frac{R^2}{(R+h)^2} = g\left(1 - 2\frac{h}{R}\right) \text{ nearly.}$$

Also it will be assumed, that within the earth the force of gravity varies directly as the distance from the centre; so that if g'' be the value of gravity at a small depth d below the earth's surface,

$$g'' = g\frac{R-d}{R} = g\left(1 - \frac{d}{R}\right).$$

PROBLEMS RELATING TO THE PENDULUM.

1. The length of the seconds' pendulum at London being 39.1393 inches, calculate the accelerating force of gravity.

2. A pendulum which beats seconds accurately on the earth's surface, loses 30 seconds in 24 hours when carried to the top of a mountain. Determine the mountain's height, supposing the earth's radius to be 3958 miles.

3. At what depth below the earth's surface will a seconds' pendulum beat only 59 times in a minute?

4. A pendulum loses 3 seconds per day; how much must it be shortened that it may beat seconds accurately?

5. Find the time of an oscillation of a pendulum 11 feet in length 3 miles above the earth's surface at the equator, where the length of the seconds' pendulum is 38.997 inches.

6. How may the pendulum be applied to determine the radius of the earth?

7. A seconds' pendulum is lengthened by 1 inch; find how many seconds it will lose in 12 hours.

8. If the pendulum be shortened 5 inches, what number of seconds will it gain in the same time?

HYDROSTATICS.

PRESSURE OF HEAVY INELASTIC FLUIDS.

1. The whole pressure on the bottom of a pail of water, the radius of which is one foot, is 120 lbs.; find the pressure referred to a unit of surface.

2. The pressure on the bottom of a vessel referred to a unit of surface is P, and it is found that 1 cubic foot of the fluid with which the vessel is filled weighs n lbs.; find the depth of the vessel.

3. An isosceles triangle is immersed perpendicularly in a fluid, with its vertex coinciding with the surface and its base parallel to it. How must it be divided by a line parallel to the base, so that the pressure upon the upper and lower parts respectively may be in the ratio of $1 : 7$?

4. A given cylinder is just immersed vertically in a given fluid; find the side of the square, upon which the pressure will be the same if it be immersed vertically with one of its sides coinciding with the surface of the fluid.

5. Determine the relation between the height and the radius of the base of a cylinder, in order that when it is just immersed vertically in a heavy fluid the pressure on the base may be equal to that on the curved surface.

6. A leaden weight is suspended by a string in a cylindrical vessel containing water; determine the additional pressure sustained by the base.

7. The sides of a hollow pyramid are isosceles triangles, the base is a rectangle having sides a and b, and the height of the pyramid is c. If the pyramid be placed with its base on a horizontal plane, and be filled with fluid, compare the pressures on the sides.

8. A cylinder is filled with fluid and laid with its axis horizontal; find the pressure on the circular base.

9. A cone is filled with fluid and laid so as to have a line in its surface horizontal; find the pressure on the circular base.

10. A given rectangle is immersed vertically in a fluid, having one side coincident with the surface. It is required to divide it by a line parallel to the surface of the fluid into two parts, the pressures on which may be in a given ratio.

11. A cylindrical vessel is filled with heavy fluid; compare the pressure on the curved surface with the weight of the fluid.

12. A hollow sphere is filled with fluid; compare the pressure on any horizontal section of the sphere with that upon any other section of the same area.

13. A hollow sphere being filled with fluid, determine those horizontal sections upon which the pressure $= \frac{3}{4}$ths the weight of the fluid. $\left(\text{Volume of sphere} = \dfrac{4\pi r^3}{3}\right)$.

14. A cylinder contains some fluid; suppose the volume of the fluid, owing to a change of temperature, to be increased by one n^{th} part, what change will take place in the pressure on the sides and base?

15. A square is immersed in a fluid, with one of its diagonals vertical; divide it by a horizontal line into two parts upon which the pressure shall be equal.

16. If an isosceles triangle be immersed in a fluid, with its base horizontal and its vertex coinciding with the surface of the fluid, how far must one side be produced in order that the pressure on the whole triangle, formed by joining its extremity with that of the other side, may be double that on the isosceles triangle?

17. A cubical vessel is filled with fluid; compare the pressures on the sides and bottom.

18. A straight line is immersed vertically in a fluid; divide it into three portions which shall be equally pressed.

19. Compare the pressures on two equal isosceles triangles just immersed in the same fluid, one with its base upwards, the other downwards.

FLOATING BODIES.

1. Supposing the specific gravity of a man, of water, and of cork, to be 1.12, 1, and .24 respectively, what quantity of cork must be attached to a man weighing 150 lbs., that he may just float in the water?

2. A rod of given length, and the density of which exceeds that of water in a given proportion, hangs over a vessel of water, having one end attached to a point, at a distance above the water less than the length of the rod, about which it can move freely in a vertical plane; determine the position of equilibrium.

3. A cylinder, the specific gravity of which is .63, floats in water, specific gravity 1; determine the portion of the cylinder immersed.

4. One fourth part of a cubical solid of given dimensions, which floats in a fluid of known specific gravity, is removed by a section parallel to the surface of the fluid, when it is found to rest with the part extant equal to twice the part before immersed; determine the weight of the solid.

5. If into a cylindrical vessel containing fluid of a given kind, a body be introduced of given weight, determine the change of pressure on the sides of the cylinder, supposing the body to float. If the body does not float, is the problem determinate?

6. A sphere of 1 foot radius, composed of matter of which the specific gravity as compared with water is .35, is

retained below the surface of water by a string; find the tension of the string. Given that the volume of a sphere $= \frac{4\pi r^3}{3}$, and that the weight of a cubic foot of water is 1000 oz.

7. A cylinder of known magnitude and specific gravity floats in water; if a small weight be placed upon the top of the cylinder find how much it will be depressed.

8. A string which is made of an elastic material, such that its length is increased 1 inch by every pound weight which is hung upon it, is made to support a sphere of known radius and specific gravity in water. Find how much the string will be stretched beyond its natural length.

SPECIFIC GRAVITY.

1. It is found that on mixing 63 pints of sulphuric acid at 1.82 specific gravity, with 24 pints of water, one pint is lost by their mutual penetration: find the specific gravity of the compound.

2. Required the weight of an hydrometer, which sinks as deep in rectified spirits, (specific gravity .866,) as it sinks in water when loaded with 67 grains.

3. Two bodies of equal weights, when connected together, will just float; what is the relation of their specific gravities and of that of the fluid?

4. If a lighter fluid rest upon a heavier, and their specific gravities be a and b, and a body, specific gravity c, rest with one part P in the upper fluid, and the other part Q in the lower, then

$$P : Q :: b - c : c - a.$$

5. The weight of P in water is 10 grains, of Q in air 14 grains, of P and Q connected together the weight in water

is 7 grains; the specific gravity of water being 1, and of air .0013, shew that the specific gravity of Q is .8237, and that it is as large as 17.023 grains of water.

6. The specific gravity of pure gold is 19.3, and of copper 8.62; required the specific gravity of standard gold, which is a mixture of eleven parts of gold and one of copper.

7. The weight of a vessel when empty being given, also when filled with water, and when filled with some other fluid, compare the specific gravity of water and of the fluid.

8. A life-boat contains 100 cubic feet of wood, specific gravity .8, and 50 feet of air, specific gravity .0013. When filled with water, what weight of iron ballast, specific gravity 7.645, must be thrown in before it will begin to sink?

9. A cylinder, placed with its axis vertical in a fluid, rests with an m^{th} part immersed; when placed in another fluid it rests with the n^{th} part immersed; to what depth would it sink in a mixture composed of equal quantities of these fluids?

10. A spherical bubble composed of matter the specific gravity of which is S, and filled with gas of the specific gravity s, just floats in air, specific gravity σ. Required the thickness of the bubble. $\left(\text{Volume of sphere} = \frac{4\pi r^3}{3}\right)$.

11. A body weighs 4oz. in vacuum, and if another body which weighs 3oz. in water be attached to it the whole in water weighs $2\frac{1}{4}$oz.; find the specific gravity of the former body.

12. If the specific gravity of air be called m, that of water being 1, and if W be the weight of any body in air, and W' its weight in water, its weight in vacuum will be

$$W + \frac{m}{1-m}(W - W').$$

13. Compare the specific gravities of two bodies, one of which weighs 10lbs. in vacuum, and the other 3lbs. in

water; the bulks of the two bodies being respectively 48 and 72 cubic inches, and the weight of a cubic foot of water 1000 oz.

14. If three fluids, the volumes of which are 3, 4, 5, and specific gravities 2, 3, 4, be mixed together, determine the specific gravity of the compound.

15. A piece of wood weighs 12 lbs., and when attached to 22 lbs. of lead and immersed in water the whole weighs 8 lbs; the specific gravity of lead being 11 times that of water, determine the specific gravity of the wood.

16. A body weighs 14 lbs. in vacuum and 9 lbs. in water; another weighs 8 lbs. in vacuum and 7 lbs. in water; compare their specific gravities.

PRESSURE OF THE AIR.

1. In an imperfectly exhausted barometer, the depression below the true altitude is to the true altitude as the space which the air left in the tube occupied before immersion is to the space which it occupies after.

2. Find the height of the mercury in a barometer when a given quantity of air is allowed to remain in the tube.

3. Supposing the pressure of atmospheric air to be 12 lbs. on the square inch, determine to what depth a piston of 3 tons weight will sink in a cylinder of radius 1 foot and height 3 feet, filled with atmospheric air.

4. A cylinder, the height of which is 6 inches, and the radius of the base one inch, is filled with atmospheric air; suppose a piston fitted into the cylinder and to be forced down through the space of 1 inch, determine the pressure of the air within the cylinder.

5. Two barometers are imperfectly filled; shew how by observations on two days to determine the quantity of air contained in each.

6. The height of the barometer at the foot of a mountain is 29.6 inches, on carrying the instrument to the summit the mercury falls 1.5 inches; taking the value of g to be 32.17, and assuming that, in the formula $p = k\rho$, $\sqrt{k} = 916.27$ feet, find the height of the mountain.

7. Given that the density of the air at the height of 7 miles is $\frac{1}{4}^{\text{th}}$ that at the earth's surface, calculate an approximate value of the quantity k in the formula $p = k\rho$.

INSTRUMENTS AND MACHINES.

1. A body when put under the receiver of a common air-pump weighs a oz. and after n turns weighs b oz. Required the weight of the body in vacuum; and supposing the specific gravity of the body known, determine the density of the air in the receiver at first.

2. What will be the number of degrees indicated by a centigrade thermometer, when Fahrenheit's stands at 50^0, 18^0, and -12^0 respectively?

3. What will be the number of degrees Fahrenheit respectively corresponding to 49^0 and -3^0 centigrade?

4. The altitude of the barometer placed in a given cylindrical diving bell is observed at the beginning and end of a descent; find the depth descended.

5. There are two air pumps, one with a receiver A and barrel B, the other with a receiver B and barrel A; compare the quantities of air exhausted by them in n turns.

6. Given the quantity of air (Q) contained in an air pump at first, it is required to determine after how many turns a given quantity (q) will be exhausted.

7. A barrel exhausts a receiver, but owing to some imperfection of the construction of the pump a given quan-

tity of common air is forced back at every stroke. Find the density of the air in the receiver after n strokes.

8. Supposing the distance from the lower valve in a common pump to be just equal to the length of a column of water which can be supported by the atmospheric pressure, find the height to which the water rises in the pump after the first stroke.

9. A cylinder of known density and magnitude, floats with its axis vertical in a vessel of water placed under the receiver of an air pump; after how many strokes of the pump will the cylinder be depressed by a given quantity?

10. A portion of the receiver of an air pump is a plane valve opening inwards and kept in its place by a spring acting with a pressure P, less than that of the atmosphere; after how many turns will the pressure of the external air open the valve?

11. In De Lisle's thermometer the boiling point is marked 0°, and the freezing point 150°. What degree of Fahrenheit corresponds to 138° of De Lisle?

12. In Reaumur's thermometer the freezing point is marked 0°, and the boiling point 80°. What degree of Reaumur corresponds to 39° Fahrenheit?

13. In Bramah's press, given the sections of the two cylinders, and the force applied to the pump, determine the pressure produced.

14. In Bramah's press, suppose the radii of the cylinders to be 2 inches and 1 foot respectively, the length of the pump handle to be three feet, and the distance of the pump from the fulcrum of the handle 4 inches, determine in what proportion the power is increased.

MISCELLANEOUS PROBLEMS.

1. Three globes of the same diameter, and of given specific gravities, are placed with their centres in the same line. How must they be disposed that they may balance on the same point of the line in vacuum and in water?

2. Explain how it is, that a ship is able to sail in a direction making an angle less than a right angle with that of the wind. What is meant by the *leeway* of a vessel?

3. Explain the construction of the sails of a wind-mill.

4. Given the weight of a body corresponding to the altitudes h and h' of the barometer, find the weight corresponding to the altitude h''.

5. Two masses of given specific gravities balance when suspended from the equal arms of a lever in a known fluid; what is the specific gravity of the fluid in which they balance when one of the masses is doubled?

6. A ship on sailing into a river sinks two inches, and after discharging 12,000 lbs. of her cargo rises one inch; find the weight of the ship and cargo.

$$\text{Given that } \frac{\text{the specific gravity of sea water}}{\text{.......................... fresh.....}} = 1.026.$$

7. Why is it necessary for a diamond merchant to have regard to the state of the weather in buying diamonds? Is it to his advantage to buy in dry or in wet weather?

8. When two persons A and B descend together to the bottom of a lake in a cylindrical diving bell, it is observed that the water stands 1 inch lower within the bell than when A descends alone; the pressure of the atmosphere is equal to that of a column of water 34 feet high, the diameter of the bell is 4 feet, and the surface of the water within it, at the bottom of the lake, is 20 feet below the surface of the lake; find the volume of B.

9. A mixture is made of two substances, first in equal quantities, then in equal weights; prove that the product of the specific gravities of the mixtures is the same as that of the specific gravities of the two substances.

10. How may the place of the barometer be supplied by weighing a body of considerable magnitude but small specific gravity?

11. A square is immersed vertically in a fluid, with one of its sides coinciding with the surface. Compare the pressures on the two triangles into which the diagonal divides it.

12. Two squares the sides of which are 4 and 2 inches respectively, are immersed vertically in a fluid, with their sides parallel to the surface. Determine to what depth the less must be immersed, so that the pressure on it may be equal to twice the pressure on the greater, the upper side of which is an inch below the surface.

13. If a cube of elastic fluid be compressed into another cube, the side of which is to the side of the former as $n : 1$; compare the whole pressure on the interior surfaces of these cubes.

14. A regular tetrahedron is filled with fluid; compare the pressure on the base with that on one of the sides.

15. A regular tetrahedron is filled with heavy fluid. Given the length of one of its edges, and the pressure on the base, find the pressure on the sides.

16. A cylindrical tube is filled with fluid and closed at both ends. Compare the pressures on its sides at the earth's surface and at a given height above it, supposing the bulk of the fluid, from change of temperature, to be diminished $\frac{1}{n}^{\text{th}}$ part, and the axis to be vertical in both cases.

17. If a cubical vessel be filled half with mercury and half with water; determine the ratio of the pressure on the sides to the pressure on the base; the density of mercury being to that of water as $m : 1$.

18. If the densities of two fluids which will not mix are in the ratio of $n : 1$, compare the quantities to be poured into a cylindrical tube of given length, so that the pressures on the concave surfaces in contact with the two fluids respectively may be in a given ratio.

19. If in a mixture of two fluids, of which the specific gravities are 3 and 5 respectively, a body of which the specific gravity is 8 loses half its weight; compare the quantities mixed.

20. Two bodies of different specific gravities balance each other in air at its mean density; if the density of the air be increased, determine which will preponderate.

21. A body is floating between two known fluids, and the part immersed in the lower is observed to be the same as if it were floating on the surface of a fluid formed by the mixture of equal quantities of the two fluids; determine the specific gravity of the solid.

OPTICS.

REFLEXION AT A PLANE SURFACE.

1. SHEW that the path of a ray of light incident from one point and reflected by a plane surface to another, is shorter than it would have been according to any other law of reflexion.

2. A ray of light is incident from a point A, and reflected at a given plane surface to another point B; supposing B fixed, find the locus of A when the whole length of the incident and reflected ray is constant.

3. A man 6 feet high stands before a vertical mirror of the same height, at a distance of 4 feet from the mirror; a lamp is placed behind him at a height of 11 feet from the ground, and at a distance of 12 feet from the mirror; find what portion of his person is illuminated, and determine his distance from the mirror when his feet are just illuminated.

REFLEXION AT A SPHERICAL SURFACE.

1. A mirror collects solar rays to a point at the distance of 6 inches from it; where will be the image of an object placed in front of it at a distance of 12 feet?

2. In reflexion at a spherical surface the conjugate foci lie on the same side of the principal focus.

3. The foci of incidence and reflexion are on opposite sides of a concave mirror of given radius, and the focus of reflexion twice as far from it as the focus of incidence. Determine their actual distances.

4. A luminous point is placed in the axis of a concave mirror of one foot radius, at the distance of 3 feet from it; find the focus of reflected rays.

5. Two parallel rays are incident on a spherical reflector at the same side of the axis; shew that the angle between the reflected rays is equal to twice the difference between the angles of incidence.

6. Find the distance of the point, to which rays diverging from a distance of 20 feet are made to converge by a concave mirror of 2 feet radius, from the principal focus of the mirror.

7. Given that the distance between the conjugate foci of a concave mirror is equal to the radius, find the focus of incidence.

8. Rays are incident upon a convex mirror of 3 feet radius from a distance of 16 feet, required the nature of the reflected pencil.

COMBINED REFLEXIONS.

1. Find the total number of images formed, when a luminous point is situated symmetrically with respect to two plane mirrors inclined at an angle of $11^0\ 15'$.

2. Determine the arrangement of a luminous point and two mirrors inclined at an angle to each other, when the images are situated in the corners of a regular hexagon.

3. At what angle must two mirrors be inclined, so that a ray incident parallel to one of them may after reflexion at each be parallel to the other?

4. If an object be placed between two parallel plane reflectors, which are moved parallel to themselves, their distance remaining constant, shew that the images formed by an even number of reflexions will remain stationary, and the other images will move in the same direction as the reflectors with twice their velocity.

5. A small pencil of rays diverges from a given point within a polished sphere, the axis of the pencil coinciding with a diameter; find the geometrical focus after two reflexions.

6. A luminous point is equidistant from two plane parallel mirrors; find the path of the axis of the small pencil of rays, by which an eye placed in a given position between the mirrors, can see the third image proceeding from either side, and shew that its length is equal to the distance of the image from the eye.

7. *BAD*, *BCE* are two plane reflectors inclined at an angle of 15°. *A* is a given luminous point in one of them. Find at what angle a ray from *A* must be incident on the other reflector, in order that after 3 reflexions it may be parallel to *BA*.

8. There are three plane reflectors, two of which are at right angles to each other, and a ray of light is incident upon the third, and reflected successively by each of them; it is required to shew that the angle between the first incident and last reflected rays is equal to twice the angle of incidence upon the first surface.

REFRACTION AT A PLANE SURFACE.

1. Find the thickness of a plane glass mirror, silvered at the back, that the distance of the image from the first surface may be twice as great as in a mirror of inconsiderable thickness.

2. At the bottom of an empty hemispherical basin a crown piece is placed, and an eye is so situated as just to see the edge of the crown piece over the rim of the basin. When the basin is filled with water the whole crown piece becomes visible. Find the radius of the basin.

3. Is it necessary to aim *above* or *below* in order to strike with a bullet a fish swimming in the water?

REFRACTION AT A SPHERICAL SURFACE.

1. A pencil of rays diverging from a point half way between the surface and the centre of a sphere, after refraction at its surface diverge from the opposite extremity of the diameter; required the refractive index.

2. If parallel rays are incident nearly perpendicularly upon a spherical refracting surface, the distance of the geometrical focus of refracted rays from the surface is to its distance from the centre as $\mu : 1$.

3. A small pencil of solar rays incident on the surface of a refracting sphere is brought to a focus upon the opposite surface of the sphere; required the refractive index of the substance of which the sphere is made.

4. A small pencil of rays is incident from a point 3 feet distant from a concave spherical surface of glass ($\mu = 1.5$), the radius of which is 2 feet; find the geometrical focus of refracted rays.

5. When divergent rays are incident from a certain point upon a spherical surface of glass, the refracted rays are found to converge to a focus at exactly the same distance on the opposite side of the surface; is the surface convex or concave? and if the position of the point of incidence be given, determine the radius of the surface.

6. There is a speck in the interior of a glass sphere; determine the apparent distance of the speck from the eye, supposing the line joining them to pass through the centre of the sphere.

REFRACTION THROUGH A PRISM.

1. A speck in the middle of the back of an isosceles prism, will appear double to an eye placed close to its edge. Suppose the angles which the two images so seen subtend at the eye to be a right angle, determine the angle of the prism.

2. If the angle of a prism be 15° 30′, and the angle of incidence 14° 18′, determine the deviation which the ray suffers in passing through the prism.

3. A ray enters a prism, the refracting angle of which is so large that no ray can pass out of it, it is therefore reflected at the second face of the prism; shew how to determine the angle between the incident and this reflected ray.

4. *ABC* is an equilateral triangle; *PQRSTV* the course of a ray refracted at *Q* and *T*, and reflected at *R* and *S*. The angle of incidence of *PQ* is 15°; find μ so that the incident and emergent rays may be inclined at an angle of 30°.

5. A ray of light is incident on a prism, in a plane perpendicular to its edge, at an angle of 45°; find the refracting angle of the prism in order that the ray may just emerge parallel to the second surface; the value of μ being $\sqrt{2}$.

6. If a ray of light be refracted through a right-angled prism in a plane perpendicular to the edge, and if $\phi\phi'$ be the angles of incidence and refraction, δ the deviation, then

$$\tan\left(\frac{\pi}{4}+\phi'\right)\cot\left(\frac{\pi}{4}+\frac{\delta}{2}\right)=\cot\left(\frac{\pi}{4}-\phi+\frac{\delta}{2}\right).$$

7. If a ray of light *QACS* be refracted through a prism *KIL* in a plane perpendicular to its edge, and if the angle of the prism $KIL = a$, $QAK = \theta$, $ACL = \phi$, and the whole deviation $= \delta$, then will

$$\tan\left(\phi-\frac{a}{2}\right)\tan\frac{a}{2}=\tan\left(\theta+\frac{\delta+a}{2}\right)\tan\frac{\delta+a}{2}.$$

8. If ϕ be the angle of incidence of a ray passing through a prism in a plane perpendicular to its edge, ψ the angle of emergence, a the angle of the prism, then

$$\sin\psi=\sin a\sqrt{\mu^2-\sin^2\phi}-\cos a\sin\phi.$$

9. A ray of light is refracted through a prism, the angle of which is 60° and index of refraction $\sqrt{2}$, in such a manner that the angles of incidence and emergence are equal, find the whole deviation. Shew also that no ray can be transmitted through a prism of the same substance when the angle exceeds 90°.

REFRACTION THROUGH A LENS.

1. An object 10 feet below the surface of water, is viewed by an eye 15 feet above the surface. What is the focal length of a lens through which it must be viewed, that its apparent depth may be 10 feet?

2. If an object is placed in the focus of a convex lens, the visual angle is the same whatever is the distance of the eye from the glass.

3. Given the radii of a thin double concave lens, upon which parallel rays are incident, find the radius of a double equiconvex lens, which compounded with the former will refract the rays parallel.

4. The back of a double convex lens is quicksilvered; if a small pencil of rays after entering the lens is reflected, find the focus of the emerging rays.

5. A small pencil of rays diverge from a point in the axis of a double convex lens, the thickness of which equals one of its radii. Required the geometrical focus of the refracted rays.

6. The concavity of a thin glass meniscus is filled with water; the radii of the surfaces are 5 and 6 inches respectively, and the refractive indices of glass and water are 1.535 and 1.336; find the focal length of the compound lens.

7. Four lenses having a common axis are placed at intervals 1, 5, 10 inches from each other, the focal length of each of the first three being 5 inches, and of the last 1 inch. If a pencil of parallel rays fall on the first, determine the point to which they will converge after passing through the system.

8*. The focal lengths of two double convex lenses are to each other as $m : n$, and the radii of their first surfaces are equal; also the radii of the surfaces of one of the lenses are as $p : q$; required the ratio of the radii of their second surfaces.

9. The radii of the surfaces of a double convex lens are 1 and 6 inches, and the refracting index 1.6; find the focal length.

10. The first surface of a lens is concave and of given radius; determine the form of the second in order that the focal length may be the same as that of a double convex lens of the same material having each of its radii equal to that of the first surface, before mentioned.

11. Determine the focal length of the lens, which causes rays diverging from a point at a distance of 2 feet to diverge as if from a distance of 3 feet.

12. What single lens is equivalent to a combination of a double convex lens of focal length 2 inches with a double concave lens of focal length 4 inches?

13. Rays diverging from a distance of 3 inches on one side of a lens are made to converge to a point 3 inches on the other side; find the focal length of the lens.

14. A double convex and a concavo-convex lens are placed in contact; the radii of the surfaces of the former are respectively 3 and 4 inches, those of the latter 3 and 5 inches, and the refractive indices of the material of the lenses are respectively 1.52 and 1.6; find the focal length of the combination.

IMAGES.

1. A straight line, .15 inches in length, is placed before a concave mirror, radius 9 inches, at a distance from the mirror equal to one third of its radius; find the magnitude and position of the image considered as a straight line.

2. A person whose eyes are 5 feet 8 inches from the ground looks into a plane vertical mirror 4 feet high; what portion of his figure will he see?

3. Given the distance of an image from a double concave lens, and the ratio of the object to the image; required the focal length.

4. How far distant must an object be placed from a plane convex lens, so that its magnitude may be n times that of its inverted image?

5. Place an object before a double convex lens, so that the image may be twice as great as the object and erect.

6. An object is placed at a distance of 4 inches from a concave lens of 2 inches focal length; is the image erect or inverted?

7. Two convex lenses, of 3 inches and 5 inches focal length respectively, are placed on the same axis at a distance of 4 inches from each other; a body is placed at a distance of 6 inches from one of the lenses, determine the position of the image formed by refraction through the combination, and whether it is erect or inverted.

VISION.

1. Given the distance at which a shortsighted person can see distinctly; determine the nature of the lens which will enable him to see at any other given distance.

2. A person can see distinctly at the distance of 2 feet, what lens will enable him to see at the distance of 24 feet?

3. A luminous point is placed between two plane mirrors inclined at a given angle; trace the course of the ray by which the point is viewed by an eye in a given position after any number of reflexions.

4. What kind of lens would be necessary in order to enable an eye to see distinctly under water?

5. A shortsighted person can see distinctly at a distance of 3 feet. He has a double concave lens, focal length 3 feet; will this enable him to see distinctly at a distance

of 12 feet? If not, find the nature of the lens, which being interposed between his eye and the former lens will be sufficient for that purpose.

6. Why cannot a person see his own image distinctly when looking into a concave reflector, his eye being between the centre of the sphere and the principal focus?

7. If a printed book is placed very near the eye, the eye is not able to distinguish the characters, but if a piece of paper pierced with a pinhole is interposed between the book and the eye it becomes possible to read, and the letters appear magnified. Explain this.

TELESCOPES.

1. Calculate the magnifying power and field of view of a Gregorian telescope from the following data:

Focal length of object-mirror = 12 inches.
.... small ...... = 1 ....
.... eye-glass = 2 ....
Breadth of eye-glass = $\frac{5}{8}$ ....

2. The focal length of the object-glass of an Astronomical Telescope is 5 feet; the eye-piece consists of two lenses of focal lengths $1\frac{1}{2}$ inch and $\frac{1}{2}$ inch respectively, separated by an interval of 1 inch; determine the distance between the object-glass and the glass nearest to it, when the telescope is adjusted.

3. Find the change in the adjustment in the preceding question necessary for a shortsighted person, who cannot see distinctly at a greater distance than 3 feet.

4. The focal lengths of the object-glass and eye-glass of a simple astronomical telescope are 27 inches and $\frac{1}{2}$ inch respectively. Find the focal length of a convex lens to be placed between them at a distance of one inch from the eye-glass, and of twice its aperture, that the field of view may be doubled, when adapted to common eyes. The field of view may be supposed to be measured by the angle sub-

tended at the centre of the object-glass by the lens nearest to it.

5. The magnifying power of an opera-glass, when directed to a distant object, is 4; but when adjusted to an object at a distance of 40 feet from the object-glass, the magnifying power is 5. Find the focal lengths of the object-glass and eye-glass.

6. An astronomical telescope consists of three convex lenses, two of which form the eye-piece; prove that if the focal length of these two be $3a$ and a respectively, and the distance between them $2a$, a real image will not be formed by the object-glass.

7. If in the preceding question the focal lengths of the lenses had been each $3a$, prove that a real image would have been formed.

8. Two astronomical telescopes, the magnifying powers of which are M and M' respectively, are placed on the same axis so as to form one telescope, find the magnifying power of the combination.

9. The focal lengths of the large and small mirror of a Gregorian telescope are f_0 and f_m respectively, the focal length of the eye-glass is f_e, and the eye-glass is fixed in a tube so as to be nearer to the eye by the distance a than the surface of the large mirror; determine the distance between the mirrors when the instrument is adjusted for an ordinary eye.

10. The magnifying power of a simple astronomical telescope is M, and the distance between the lenses when the instrument is adjusted for an ordinary eye is D; find the focal lengths of the lenses.

11. In using a simple astronomical telescope with a certain eye-glass it is found that the magnifying power is only half what is required; a new eye-glass, the focal length of which is 1 inch less than that of the former, is therefore substituted, and it is found that the magnifying power is now exactly that required, namely 10; find the focal lengths of the lenses.

MISCELLANEOUS PROBLEMS.

1. Two fixed points, situated on the same side of an indefinite straight line, are viewed from different points of that line; determine by geometrical construction the point at which their apparent distance from each other is greatest.

2. Find the point in a reflecting circular arc, at which a ray parallel to the axis must be incident, in order that after reflexion it may pass through a given point of the circumference.

3. If a plane mirror revolve about an axis in its plane, the angular motion of the image of a fixed straight line will be double that of the mirror.

4. A concave hemispherical mirror is filled with water; given the focus of rays diverging upon the surface of the water from a point upon the axis of the mirror, find the focus of the rays which emerge after one reflexion and two refractions.

5. A ray of light is incident from a luminous point, at an angle the tangent of which is $\frac{4}{\sqrt{5}}$, on a plate of glass ($\mu = 1.5$) the thickness of which is t; find the point in which the direction of the ray, after passing through the glass, cuts the line drawn from the luminous point perpendicular to the glass. Find also the point in which the ray would cut the same line if the angle of incidence were indefinitely small, and shew that the distance between the two points is $\frac{4}{15}t$.

6. A ray of light, issuing from a point in the extreme ordinate of a parabola, is incident in a direction parallel to the axis and after two reflexions at the curve meets the ordinate again; prove that the length of the path described will be the same from whatever point in the ordinate the ray proceeds.

7. Find the point at which a ray of light must be incident parallel to the axis upon a concave spherical reflector, that after two reflexions it may cut the axis at a given angle.

8. Where must a ray of light parallel to the axis of a concave spherical reflector be incident, that after reflexion it may divide the radius in the proportion of $\sqrt{3} - 1 : 1$?

9. What will be the effect produced upon the appearance of objects by viewing them through the wrong end of a telescope?

10. The ends of a glass cylinder are worked into portions of a convex and concave spherical surface, the radii being r and s respectively, and their centres being in the axis of the cylinder; shew that the distance of these surfaces, in order that an eye placed at the concave surface may see the image of a distant object distinctly, must be $\dfrac{\mu(r-s)}{\mu-1}$, and that the magnifying power $= \dfrac{r}{s}$.

11. Rays issuing from a luminous point are incident upon a thin lens. A portion of those which enter the lens proceeds at once through the second surface; a second portion does not escape till it has been twice internally reflected; a third portion four times reflected; a fourth portion six times, and so on. Shew that a row of images will be formed at distances from the lens which are in harmonical progression.

12. A ray of light falls on the convex surface of a hemispherical lens in a direction parallel to its axis, is reflected at the plane surface, and emerges through the convex; prove that the angles of incidence and emergence are equal.

13. A double convex lens is silvered at the back, and a small pencil of rays diverging upon it enter, and after reflexion at the silvered surface emerge by refraction; find the geometrical focus of the emergent rays.

14. A small pencil of rays is refracted through a sphere, find the geometrical focus of emergent rays.

15. Prove that a sphere has the effect of a convex lens, and find the radius of a glass sphere which shall be equivalent to a double convex lens of 3 inches focal length.

16. A spectator on the bank of a river perceived that the top of a tree, which stood on the other bank directly opposite, was reflected from a point of the water 6 feet from the place where he stood; and that after walking 20 yards along the bank, the distance of the new point of reflexion from him was then 8 feet; given that his eye is 5 feet above the surface of the water, find the height of the tree.

The Book of Solutions end here

APPENDIX,

Containing the QUESTIONS proposed in several years during the first three days of the SENATE-HOUSE EXAMINATION, in accordance with the *Grace* of the SENATE passed in *May* 1846.

1848.

MODERATORS.

GEORGE GABRIEL STOKES, M.A., Pembroke College.
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EXAMINERS.

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THURSDAY, *Jan.* 6. 9...12.

1. IF two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, the angle which is contained by the two sides of the one shall be equal to the angle contained by the two sides, equal to them, of the other.

How does it appear that the two triangles are equiangular and equal to each other?

2. The opposite sides as well as the opposite angles of a parallelogram are equal to one another, and the diameter bisects it.

If the two diameters be drawn, shew that a parallelogram will be divided into four equal parts. In what case will the diameter bisect the angle of a parallelogram?

3. If a straight line be divided into two equal parts, and also into two unequal parts, the rectangle contained by the unequal parts, together with the square of the line between the points of section, is equal to the square of half the line.

4. Equal straight lines in a circle are equally distant from the centre; and conversely, those which are equally distant from the centre are equal to one another.

Shew that all equal straight lines in a circle will be touched by another circle.

5. The angle at the centre of a circle is double of the angle at the circumference upon the same base, that is, upon the same part of the circumference.

If two straight lines AEB, CED in a circle intersect in E, the angles subtended by AC and BD at the centre are together double of the angle AEC.

6. Describe an isosceles triangle having each angle at the base double of the third angle.

7. If the first of six magnitudes have to the second the same ratio which the third has to the fourth, but the third to the fourth a greater ratio than the fifth to the sixth, the first shall have to the second a greater ratio than the fifth has to the sixth.

8. Equal triangles which have one angle of the one equal to one angle of the other have the sides about the equal angles reciprocally proportional; and conversely.

9. If two planes which cut each other be each of them perpendicular to a third plane, their common section shall be perpendicular to the same plane.

10. Define a parabola, and its tangent.

Assuming the tangent at any point P of a parabola to make equal angles with the focal distance SP and the diameter at that point, prove that SY, the perpendicular upon it from the focus, meets it in the tangent at the vertex.

If PM be the ordinate at P, and T the intersection of the tangent at P with the axis, $TP \cdot TY = TM \cdot TS$.

11. Prove that in a parabola $QV^2 = 4SP \cdot PV$.

Define the parameter at any point of a parabola, and prove that it is proportional to the focal distance of the point.

12. Define an ellipse. If one of the focal distances SP of a point P be produced to L, a straight line PT which bisects the exterior angle HPL is the tangent to the curve at P.

For what position of P is the angle SPH greatest?

13. State when diameters of an ellipse or hyperbola are conjugate. Prove that all parallelograms whose sides touch an ellipse at the ends of conjugate diameters are equal.

Prove that such parallelograms have the least area of all which circumscribe the ellipse.

14. In an ellipse the sum of the squares of any two conjugate diameters is invariable.

When is the square of their sum least?

15. Define the asymptotes of an hyperbola. If any straight line Qq perpendicular to either axis of an hyperbola meet the asymptotes in Q, q, and the curve in P, the rectangle $QP.Pq$ is invariable.

16. If a circular right cone be cut by a plane which meets both of its slant sides, the section is an ellipse.

17. The chord of curvature of an ellipse or hyperbola through its centre is equal to $\frac{2CD^2}{CP}$.

The chords of curvature through the centre and focus are in the ratio of AC to CP.

THURSDAY, *Jan.* 6. $1\frac{1}{2}...4$.

1. PROVE the rules for finding the greatest common measure and least common multiple of two integers.

Find the least number of pounds which can be paid in either half-crowns or guineas.

2. Explain the meaning and use of fractions in a system of arithmetic, and shew that the value of a fraction remains unchanged when its two members are replaced by any equimultiples of their former values.

3. If each inmate of a workhouse cost per week, 2*s.* $2\frac{1}{4}$*d.* for food, $5\frac{1}{4}$*d.* for clothes and washing, and 3*d.* for lodging, and the number of persons thus maintained be estimated at 100000, what is the whole yearly cost; and how many labourers at 18*d.* a day wages would earn the same sum in the same time?

4. If 25 men do a piece of work in 24 days, working 8 hours a day, in how many days would 30 men do the same piece of work working 10 hours a day?

5. Supposing arithmetical addition and subtraction represented by the signs +, − respectively, prove the equivalence of the two sets of operations indicated by

$$(a-b)(c-d) \text{ and } ac+bd-bc-ad,$$

a being a number greater than b, c greater than d.

What is the nature of the generalization with respect to the use of the signs +, − in Symbolical Algebra?

6. Find the highest common divisor of

$$x^3-x^2-2x+2 \text{ and } x^4-3x^3+2x^2+x-1.$$

7. Define what is meant by the n^{th} root of a given numerical magnitude, and explain the principle according to which a root is represented by means of a fractional index.

8. Define a logarithm, and the base of a system of logarithms.

Prove that $\log xy = \log x + \log y$, $\log x^n = n \log x$.

Given
$$\log 7 = \cdot 8450980,$$
$$\log 58751 = 4\cdot 7690153,$$
$$\log 58752 = 4\cdot 7690227,$$
find $\sqrt[5]{\cdot 07}$ to 7 significant figures.

9. Solve the equations,

(1) $$\frac{x+1}{x-1} + \frac{x+2}{x-2} = 2\frac{x+3}{x-3}.$$

(2) $$\sqrt{1+x+x^2} - \sqrt{1-x+x^2} = mx.$$

(3) $$x+y-z = 8x+3y-6z = 3z-4x-y = 1.$$

Discuss the nature of the roots in (2) as affected by the value of m.

10. When is one quantity said to vary as another, directly or inversely? If $A \propto B$ when C is constant, and $A \propto C$ when B is constant, prove that $A \propto BC$ when B and C both vary.

Given that the area of an ellipse varies as either axis when the other is constant, and that the area of a circle of radius unity $= 3.14\ldots$, find the area of the ellipse whose axes are 3 and 5.

11. Find the sum to n terms of a geometric series. What is meant by the sum of an infinite series? When can such a series be said to have a sum?

If P be the sum of the series formed by taking the 1st and every p^{th} term of an infinite geometric series whose first term is 1 and whose ratio < 1, Q the sum of the series found by taking the 1st and every q^{th} term, prove that

$$P^q(Q-1)^p = Q^p(P-1)^q.$$

12. Assuming the Binomial Theorem in the case of positive integral indices, prove it in the case of negative and fractional indices.

Write down the general term of the expansion of

$$(1 - 2x)^{-\frac{7}{2}}.$$

13. Trace the change in sign and magnitude of the tangent and secant of an angle through the first four quadrants.

14. Prove that

$$\sin (A - B) = \sin A . \cos B - \cos A . \sin B.$$

Give a general enunciation for the construction requisite in order to obtain the above expression for all values of A and B.

15. Express $\sin 2A$ in terms of $\tan A$.

Given $\tan \frac{A}{2} = 2 - \sqrt{3}$, find $\sin A$, and thence A.

16. Prove that the sines of the angles of a triangle are proportional to the opposite sides.

Hence deduce the expression for the cosine of an angle in terms of the sides.

17. Express the area of a triangle in terms (1) of two sides and the contained angle, (2) of one side and the adjacent angles.

Two sides of a triangle are equal to 3 and 12 respectively, and the contained angle is equal to 30^0; find the hypotenuse of an equal right-angled isosceles triangle.

18. Shew how to determine the height of a mountain by observations at two stations in the same horizontal plane, the distance between the stations being known.

If the stations are in the same vertical plane passing through the summit, and the summit (S) is observed from the further station, but a lower point (S') is observed by mistake from the nearer, shew that the height determined by the process lies between the heights of S and S'.

FRIDAY, *Jan.* 7. 9...12.

1. IF two forces, acting on a particle, be represented by two adjacent sides of a parallelogram, prove that their resultant will act in the direction of the corresponding diagonal, pointing out any assumptions or propositions which you may employ in the proof.

Explain how the force of the current may be taken advantage of to urge a ferry-boat across a river, the centre of the boat being attached, by means of a long rope, to a mooring in the middle of the stream.

2. When a weight is supported on a smooth inclined plane by a force along the plane, the force is to the weight as the height of the plane is to its length.

If the roughness of a plane, which is inclined to the horizon at a known angle, be such that a body will just rest supported on it, find the least force along the plane requisite to drag the body up.

3. Find the relation of P to W in the system of pullies where each string is attached to the weight; and prove that $Pp = Ww$, where p, w, are the spaces gone through by P and W respectively when the system is put in motion.

4. When a body is kept in equilibrium by three forces acting in one plane, either their directions are parallel, and one force is equal to the sum or difference of the other two, or their directions meet in a point, and each force is as the sine of the angle between the other two.

AB is a rod capable of turning freely about its extremity A, which is fixed, CD is another rod equal to $2AB$ and attached at its middle point to the extremity B of the former, so as to turn freely about this point; a given force acts at C in the direction CA, find the force which must be applied at D in order to produce equilibrium.

5. Assuming the principle of the straight lever for two forces, find the condition of equilibrium of a rigid

body moveable about a fixed axis, and acted on by any number of forces in a plane perpendicular to the axis.

If a set of forces, acting at the angular points of a plane polygon, be represented by the sides, taken in order, shew that their tendency to turn a body about an axis perpendicular to the plane of the polygon is the same, through whatever point of the plane the axis passes.

6. Prove that the statical effect due to the weights of the several particles of which a body is composed is the same as it would be if all the matter were collected at its centre of gravity.

Shew that the centre of gravity of a triangular area coincides with that of three particles of equal weight placed at the angular points, and thence deduce its position.

7. Enunciate the first and second laws of motion, and mention experimental facts which would lead to their assumption.

What is the nature of the final evidence which is considered conclusive as to the truth of these laws?

8. Shew that in uniformly accelerated motion

$$s = \tfrac{1}{2}ft^2,$$

proving, if your method require it, but not assuming, that if the velocity of the body be reversed the backward motion will be exactly similar to the forward motion.

9. The time of descent down any chord passing through the highest or lowest point of a vertical circle is the same as the time down the vertical diameter.

10. Prove that a body projected obliquely and acted on by gravity will describe a parabola.

Find the velocity and direction of projection, in order that the projectile may pass horizontally through a given point.

11. State and explain the third law of motion.

Can we form a conception of mass without introducing the idea of weight?

12. A body, whose mass is m, is projected with a velocity $\dot{V}$, and acted on by a constant pressure P in the line of projection, find the velocity of the body at any time; and if the pressure act in a direction opposite to that of projection, find how long it will be before the body is brought to rest.

A train of connected bodies, whose weights are W_1, W_2,..., are moving together in a straight line, being acted on by the retarding pressures P_1, P_2,... respectively, find the conditions in order that the bodies may continue to move with equal velocities when the connexion between them is severed.

13. Find the accelerating force when one weight pulls another over a fixed pulley. Find also the tension of the string.

14. If two perfectly elastic balls moving with given velocities in a straight line impinge directly, find their velocities after impact.

If the first, A, of three perfectly elastic balls placed in a line impinge directly with a given velocity on the second B, so that B in turn impinges on the third C, find the mass of B in order that the velocity given to C may be the greatest possible, the masses of A and C being known.

15. Find the time of oscillation of a body oscillating in a cycloid.

FRIDAY, *Jan.* 7. $1\frac{1}{2}$...4.

1. DEFINE a fluid, and prove that a pressure applied to the surface of a fluid mass in equilibrium will be equally transmitted in all directions.

2. In a mass of homogeneous liquid in equilibrium under the action of gravity, the difference of pressure at two points in the same vertical line which falls wholly within the fluid is proportional to the distance between those points.

What further considerations are necessary to prove this proposition when the straight line joining the two points in question does not fall wholly within the fluid?

3. When a liquid is in equilibrium in a vessel, prove that the vertical component of the pressure on any portion of the surface beneath is equal to the weight of the superincumbent column of fluid, supposing the column to reach to the surface without interruption.

How must the enunciation be altered so as to make the proposition apply (1) to the case in which the column is interrupted; (2) to the case in which the surface considered is pressed upwards?

4. When a solid floats in a liquid, its weight is equal to that of the fluid displaced, and its centre of gravity is in the same vertical line with that of the fluid displaced.

Is this statement true in the case of a body partly immersed in several liquids which are superposed?

5. Describe the experiment which shews that the pressure of air is proportional to its density while the temperature remains constant. State the relation between the pressure, density, and temperature of air.

A straight vertical tube is closed at its lower end; how much of a given liquid can be poured into it, the air which originally filled it being compressed at the bottom of the tube?

6. Describe the nature and use of the Diving Bell. Does the tension of the rope by which it is suspended increase or decrease as the bell is sunk lower?

7. Describe the Siphon, and its action. What would be the effect of opening an orifice in the highest point of the tube?

8. Define the specific gravity of a substance; and shew how the specific gravity of a solid may be found by weighing it in air and water. How is the method modified if the solid floats in water?

9. Describe the construction and action of the Condensing Steam Engine. In what respect was this an improvement upon the Atmospheric Engine?

10. Define a pencil of rays, converging rays, diverging rays, and the focus of a pencil of rays.

If diverging or converging rays be reflected at a plane surface, the foci of incident and reflected rays are on contrary sides of the reflector, and equally distant from it.

Why does a common looking-glass give more than one image of a point?

11. Define conjugate foci, and shew that, in the case of reflection at a spherical surface, they lie on the same side of the principal focus, that they move in opposite directions, and meet at the centre and surface of the reflector.

12. A ray of light cannot pass out of a denser into a rarer medium if the angle of incidence exceeds a certain limit.

Explain the appearance which will be presented to an eye placed under water and looking upwards.

13. Parallel rays, refracted at a plane surface, continue parallel.

14. Parallel rays, refracted at a convex spherical surface of a denser, or a concave of a rarer medium, into which they pass, are made to converge; and refracted at a concave spherical surface of a denser, or a convex of a rarer medium are made to diverge.

15. If A, a be two points on the surfaces of a lens where the radii are parallel to each other, the incident and emergent parts of a ray of light which passes through the lens in the direction Aa will be parallel.

Define the centre of a lens, and shew that it is a fixed point.

16. The image of a straight line formed by a plane refracting surface is a straight line.

Why does a straight rod appear bent when partly immersed in water? What must be its inclination to the horizon when its apparent portions are inclined to each other at the greatest angle?

17. Describe the Astronomical Telescope; trace the course of a pencil of rays from any point of a distant object, and find the magnifying power.

If the focal lengths of the lenses be 12 inches and 1 inch, how far must the eye-glass be moved for viewing an object at a distance of 40 feet from the object-glass.

SATURDAY, *Jan.* 8. 9...12.

1. ENUNCIATE and prove Newton's first Lemma. In what Lemma is the idea of the ultimate ratio of two vanishing quantities first introduced?

Give Newton's answer to the objection, "that there can be no ultimate ratio of vanishing quantities, since their ratio cannot be considered ultimate before they have vanished, and after they have vanished there can be no ratio at all."

2. State and prove Lemma VII. Draw a figure representing an intermediate state of the construction in this Lemma when the point B has moved up into a position nearer to A.

3. State and prove Lemma X, and shew that under the circumstances of the enunciation,

$$\text{force} = 2 \text{ limit} \frac{\text{space}}{(\text{time})^2}.$$

4. If a body revolve about a fixed centre of force, the areas described by lines drawn from the body to the centre of force lie in one plane and are proportional to the times of describing them.

Point out the laws of motion assumed in the proof of this proposition.

5. The centripetal forces of bodies which describe different circles with uniform velocities tend to the centres of the circles, and are to each other directly as the squares of the arcs described in the same time, and inversely as the radii of the circles.

How much must the length of the day be shortened, in order that the rotation of the Earth may be sufficiently rapid to destroy the weight of bodies at the equator?

6. A body describes an ellipse; required the law of force tending to the centre.

Give the reasoning by which Newton extends the result obtained to the case of a parabola, or an hyperbola. Is the same kind of reasoning applicable when the centre of force is at the focus, instead of at the centre?

7. Find the law of force tending to the focus of a parabola.

If the latus rectum of a parabola is 24 feet, and the velocity of a body revolving in it at the vertex is 2 yards per minute, find the time in which the body moves from the vertex to one end of the latus rectum.

8. State Kepler's laws; and enunciate the various propositions in Newton by means of which they may be deduced from the theory of universal gravitation.

How does this theory account for the deviations from the exactness of these laws which accurate observations have shewn to exist?

9. Describe the phenomenon of the Sun's apparent motion among the stars, and mention the two principal hypotheses which have been made to account for it.

What are the chief arguments in favour of the received hypothesis, that the Earth moves round the Sun in an orbit inclined to the plane of the equator?

When does the Sun set at the point of the horizon opposite to that at which it rose? $\frac{1}{2}$ a year later.

10. Explain the kinds of observations which a transit, a mural circle, and an equatoreal are respectively designed to make. Describe the process of finding the error and rate of a clock at an observatory.

11. Distinguish between sidereal time, solar time, and mean solar time; and explain why the time indicated by the common clock is sometimes before, sometimes behind that indicated by the dial.

The equation of time at noon on one day is $3^m\ 14^s$, and at the succeeding noon is $3^m\ 12^s$, what time ought a correct watch to shew when a sun-dial marks 6 o'clock on the evening of the former day?

12. Give an explanation of the phenomenon of refraction, and point out the astronomical observations, made in the plane of the meridian, which are affected by it.

What is the cause of twilight; and why is its duration so mueh less in the tropics than in the higher latitudes?

13. What is meant by the Precession of the Equinoxes; how is its existence manifested, and what is the physical cause of it?

14. Account for the phases of the Moon in the course of a month, and shew how a lunar eclipse arises.

Why are the satellites of Jupiter more frequently eclipsed than the Moon?

15. Explain how the finite velocity of light causes a difference between the real and apparent places of a fixed star. What additional consideration arises in the case of a planet?

16. Shew how the Moon's motion among the stars is made to determine the longitude at sea.

SATURDAY, *Jan.* 8. 1...4.

1. If the hypotenuse AB of a right-angled triangle ABC be bisected in D, and EDF drawn perpendicular to AB, and DE, DF cut off each equal to DA, and CE, CF joined, prove that the last two lines will bisect the angle at C and its supplement respectively.

2. A, B, C are three given points in the circumference of a given circle; find a point P such that if AP, BP, CP meet the circumference in D, E, F, the arcs DE, EF may be equal to given arcs.

3. The angles of a quadrilateral inscribed in a circle, taken in order, when multiplied by 1, 2, 2, 3 respectively, are in arithmetical progression, find their values.

4. A ship sails with a supply of biscuit for 60 days, at a daily allowance of 1 lb. a head; after being at sea 20 days she encounters a storm, in which 5 men are washed overboard, and damage sustained that will cause a delay of 24 days, and it is found that each man's allowance must be reduced to $\frac{5}{7}$ lb.; find the original number of the crew.

5. If a, b, and x be positive, and $a > b$, prove that

$$\frac{x+a}{\sqrt{(x^2+a^2)}} >< \frac{x+b}{\sqrt{(x^2+b^2)}} \text{ according as } x >< \sqrt{ab}.$$

6. If a, b, c be in harmonic progression, and n be a positive integer, shew that $a^n + c^n > 2b^n$.

7. Prove that $\sin 3\theta \sin^3\theta + \cos 3\theta \cos^3\theta = \cos^3 2\theta$.

8. Having given the three right lines drawn from any point to the three angular points of an equilateral triangle, determine a side of the triangle.

9. Given the lengths of the axes of an ellipse, and the positions of one focus and of one point in the curve, give a geometrical construction for finding the centre.

10. P is any point in an ellipse, AA' its axis major, NP an ordinate to the point P; to any point Q in the curve draw AQ, $A'Q$ meeting NP in R and S; shew that

$$NR \cdot NS = NP^2.$$

11. PSp is a focal chord of a parabola, RDr the directrix, meeting the axis in D; Q is any point in the curve; prove that if QP, Qp produced meet the directrix in R, r, half the latus rectum will be a mean proportional between DR, Dr.

12. Two bodies acted upon by gravity are projected obliquely from two given points, in given directions and with given velocities; determine their position when their distance is the least possible.

13. A railway train is going smoothly along a curve of 500 yards radius at the rate of 30 miles an hour; find at what angle a plumb-line hanging in one of the carriages will be inclined to the vertical.

14. If a body be projected from a given point in a given direction with a given velocity, and be acted on by a force tending to S and varying as $\frac{1}{(\text{dist.})^2}$, prove that if PSp be any focal chord of the body's path the sum of the squares of the velocities at P and p will be constant.

15. A number of balls of given elasticity $A, B, C\ldots$ are placed in a line; A is projected with a given velocity so as to impinge on B; B then impinges on C, and so on; find the masses of the balls $B, C\ldots$ in order that each of the balls $A, B, C\ldots$ may be at rest after impinging on the next; and find the velocity of the n^{th} ball after its impact with the $(n-1)^{\text{th}}$.

16. An imperfectly elastic ball is projected in a given direction within a fixed horizontal hoop, so as to go on rebounding from the surface of the hoop; find the limit to which the velocity of the ball will approach; and shew that it will attain this limit at the end of a finite time.

17. If Q, q be two points in the radius of a spherical refracting surface whose centre is E, such that $EQ : Eq ::$ the sine of incidence : the sine of refraction, determine geometrically the position of the point P so that a ray proceeding from Q and incident upon the surface at P may after refraction proceed from q.

18. If a ray of light after being reflected any number of times in one plane at any number of plane surfaces return on its former course, prove that the same will be true of any ray parallel to the former which is reflected at the same surfaces in the same order, provided the number of reflections be even.

19. An inverted vessel formed of a substance which is heavier than water contains enough of air to make it float; prove that if it be pushed down through a certain space, it will be in a position of unstable equilibrium; and determine the space in question.

20. A uniform piston, terminated by a plane of area A perpendicular to its side, is inserted into an orifice in a vessel containing fluid; prove that the work done in gently pushing in the piston through a small space s is ultimately equal to the work done in lifting a portion of the fluid of volume As through a height equal to the depth of the centre of gravity of the plane below the surface of the fluid.

21. Two equal slender rods AB, AC moveable about a hinge at A and connected by a string BC rest with the angle A immersed in a given fluid; determine the tension of the string BC.

22. If a rectangular court be enclosed within a wall of given height, and one of its sides be inclined at an angle of 30° to the meridian, determine the breadths of the shadows of the walls on a given day at noon, and the portions of the courts and walls which will be enveloped in the shadow, the latitude being $52^\circ\ 30'$ N., and the Sun's declination on the given day $7^\circ\ 30'$ N.

1849.

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THURSDAY, *Jan.* 4. 9...12.

1. DESCRIBE an equilateral triangle upon a given finite straight line.

By a method similar to that used in this problem, describe on a given finite straight line an isosceles triangle, the sides of which shall be each equal to twice the base.

2. If a side of any triangle be produced, the exterior angle is equal to the sum of the two interior and opposite angles; and the three interior angles of every triangle are together equal to two right angles.

Can you give Legendre's method of demonstrating this proposition, which depends upon the necessary homogeneity of algebraical equations, or any demonstration other than Euclid's?

3. Divide a given straight line into two parts, so that the rectangle contained by the whole and one of the parts shall be equal to the square of the other part.

Shew that in Euclid's figure four other lines, beside the given line, are divided in the required manner.

4. If a straight line touch a circle, the straight line drawn from the centre to the point of contact shall be perpendicular to the line touching the circle.

Give a direct demonstration of this proposition by the method of limits.

5. Inscribe a circle in a given triangle.

How may a circle be described touching one side and the produced parts of the other two?

6. If any number of magnitudes be proportionals, as one of the antecedents is to its consequent, so shall all the antecedents taken together be to all the consequents.

What restriction is here implied as to the species of the magnitudes?

7. The sides about the equal angles of equiangular triangles are proportionals, and those sides which are opposite to the equal angles are homologous.

Apply this proposition to prove that the rectangle contained by the segments of any chord passing through a given point within a circle is constant.

8. Define compound ratio; and prove that equiangular parallelograms have to each other the ratio which is compounded of the ratios of their sides.

Of what use is this proposition in the application of Algebra to Geometry?

9. Draw a straight line perpendicular to a plane from a given point without it.

Prove that equal right lines drawn from a given point to a given plane are equally inclined to the plane.

10. In the parabola, the rectangle under the latus rectum and an abscissa of the axis is equal to the square of the semi-ordinate.

11. The normal at any point of an ellipse bisects the angle between the focal distances.

Can you deduce the proof of this proposition from mechanical considerations?

12. The perpendiculars from the foci on the tangent to an ellipse intersect the tangent in the circumference of a circle having the axis major as diameter.

Deduce from this an analogous proposition for the parabola.

13. In the ellipse, if the conjugate diameter meet either focal distance in E, PE will be equal to AC.

14. Define the circle of curvature; and prove that in the ellipse the diameter, the conjugate diameter, and the chord of curvature passing through the centre, are in continued proportion.

15. If a tangent be drawn to a hyperbola, and be terminated by the asymptotes, it will be bisected in the point of contact.

Apply this proposition to prove directly that the area of the triangle contained by the tangent and the asymptotes is constant.

If SVs, TVt be two tangents cutting one asymptote in the points S, T, and the other in s, t, prove that

$$VS : Vs :: Vt : VT.$$

16. The section of a right cone by a plane parallel to a line in its surface, and perpendicular to the plane containing that line and the axis, is a parabola.

The foci of all parabolic sections which can be cut from a given right cone lie upon the surface of another cone.

THURSDAY, *Jan.* 4. $1\frac{1}{2}\ldots 4$.

1. Shew that the value of a product does not depend upon the order of its factors, the factors being commensurable. Extend your proof to the case of incommensurable factors.

2. A person rents a piece of land for £120 a year. He lays out £625 in buying 50 bullocks. At the end of the year he sells them, having expended £12. 10*s*. in labour. How much per head must he gain by them, in order to realize his rent and expenses, and 10 per cent. upon his original outlay?

3. A person in London owes another in Petersburgh a debt of 460 rubles, which must be remitted through Paris. He pays the requisite sum to his broker, at a time when the exchange between London and Paris is 23 francs for £1, and between Paris and Petersburgh 2 francs for one ruble. The remittance is delayed until the rates of exchange are 24 francs for £1, and 3 francs for 2 rubles. What does the broker gain or lose by the transaction?

4. Prove the rules for the multiplication and division of decimals. Shew that every vulgar fraction must produce either a terminating or a recurring decimal.

If $\frac{a}{b}$ be a fraction, and if a and the successive remainders be multiplied by a series of quantities q, q', q'', &c., and the successive products be divided by b, giving quotients p, p', p'', &c., shew that

$$\frac{a}{b} = \frac{p}{q} + \frac{p'}{qq'} + \frac{p''}{qq'q''} + \&c.$$

5. Prove the rule for extracting the square root of an integer.

6. Assuming generally that $a^m . a^n = a^{m+n}$, determine the meaning which must be assigned to a^m when m is fractional or negative.

7. Solve the equations:

(1) $$(a - x)\frac{x + m}{x + n} = (a + x)\frac{x - m}{x - n}.$$

(2) $$x^4 + 4a^3x = a^4.$$

(3) $$\begin{cases} x^2 + y^2 = a^2, \\ x \cos \alpha + y \sin \alpha = a \sin \beta. \end{cases}$$

8. Shew that a ratio of greater inequality is diminished by adding the same quantity to both terms.

9. Find the number of combinations of n things taken r together; and shew that the number of combinations of

n things taken r together is equal to the number taken $n-r$ together.

There are n points in a plane, no three of which are in the same straight line, with the exception of p, which are all in the same straight line; find the number of lines which result from joining them.

10. Define the cosine of an angle; and trace its changes in sign and magnitude as the angle increases from 135^0 to 405^0.

Construct the angle whose tangent is $3-\sqrt{2}$.

11. Express $\sin A$ in terms of $\sin 2A$; explain why there are four values, and how the correct one is to be selected.

Ex. Find $\sin 9^0$ and $\sin 81^0$.

12. Two sides and the included angle of a triangle being given, shew how to find the remaining angles.

The ratio of two sides of a triangle is $9:7$, and the included angle is $47^0\ 25'$, find the other angles.

Given $\log 2 = \cdot 3010300$,

$\text{L}\tan 66^0\ 17'\ 30'' = 10\cdot 3573942$,

$\text{L}\tan 15^0\ 53' = 9\cdot 4541479$; diff. $1' = 4797$.

13. Find the area of a quadrilateral figure, whose opposite angles are supplementary, in terms of its sides.

If the sides taken in order are $3, 3, 4, 4$, find the area, and the radii of the inscribed and circumscribed circles.

14. Shew how to find the height of an object above a horizontal plane, from observations made at two given stations in the plane.

The angular elevation of a tower at a place A due south of it is 30^0, and at a place B, due west of A, and at the distance a from it, the elevation is 18^0; shew that the height of the tower is $\dfrac{a}{\sqrt{2\sqrt{5}+2}}$.

15. Define a logarithm; and find that of 256 to the base $2\sqrt{2}$.

What is the advantage of taking 10 for the base of a system of logarithms?

Calculate a table of proportional parts for every $10''$, corresponding to a tabular difference of 4797 for $1'$.

FRIDAY, *Jan.* 5. 9...12.

1. Assuming the Parallelogram of Forces, so far as the direction of the resultant is concerned, shew that the diagonal of the parallelogram represents the magnitude of the resultant.

The resultant of two forces is 10lbs., one of them is equal to 8lbs., and the direction of the other is inclined to the resultant at an angle of 36°. Find the other force, and the angle between the two.

2. Two forces act in the same plane at given points in a rigid body; if any point of a certain straight line be fixed, shew that there will be equilibrium. Also if perpendiculars be drawn from this point upon the directions of the forces, the forces will be to each other inversely as these perpendiculars.

3. Find the relation between P and W in the system of pullies in which the same string passes round all the pullies.

A triangular plane ABC is kept in equilibrium by three systems of pullies of the above kind, each having one block fastened to a fixed external point and the other attached to an angular point of the triangle by a string whose direction bisects the angle. The same string passes round all the pullies and is solicited by a certain force. Shew that the numbers of the strings between the pullies are as $\cos\frac{A}{2} : \cos\frac{B}{2} : \cos\frac{C}{2}$.

4. State and explain the conditions necessary and sufficient for the equilibrium of a body which has one or more points in contact with a smooth plane, and is acted upon by any forces.

A triangular board of given weight rests in equilibrium with its base on a horizontal plane sufficiently rough to prevent all sliding. A force acts upon it in its own plane and in a given line drawn through the vertex and without the triangle; find by a geometrical construction, or otherwise, the limits between which the magnitude of the force must lie if the equilibrium is preserved.

5. Explain the nature of the action and reaction of smooth surfaces in contact.

Two equal circular disks with smooth edges, placed on their flat sides in the corner between two smooth vertical planes inclined at a given angle, touch each other in the line bisecting the angle. Find the radius of the least disk which may be pressed between them without causing them to separate.

6. Find the relation between the power and the weight upon the Screw; and shew that

$$\frac{P}{W} = \frac{W\text{'s virtual velocity}}{P\text{'s virtual velocity}}.$$

7. Find the centre of gravity of a triangle.

One corner of a triangle, equal to $\frac{1}{n}$th part of its area, is cut off by a line parallel to its base; find the centre of gravity of the remainder.

8. State the second law of motion and explain its use; and mention some experiments which give results in accordance with it.

9. Distinguish between *accelerating* and *moving* force. Explain what is meant by the phrase "action and reaction are equal and opposite."

10. If a body be projected with a velocity u, and acted on by a uniform force f in the direction of motion, shew that the space passed over in the time t will be $ut + \frac{1}{2}ft^2$.

A particle moves over 7 feet in the first second of the time during which it is observed, and over 11 and 17 feet in the third and sixth seconds respectively. Is this consistent with the supposition of its being subject to the action of a uniform force?

11. Two equal balls A and B are moving with given velocities in the same plane in directions at right angles to each other, and the line joining their centres at the instant of impact is in the direction of A's motion. Determine their motions after impact, supposing them smooth and inelastic.

12. A heavy particle is drawn up an inclined plane by means of another attached to it by a string passing over the upper edge of the plane, the latter particle descending vertically; find the accelerating force.

Find also the tension of the string; and state what would take place if the string were cut at any instant.

13. The velocity of a projectile at any point of its parabolic path is that which would be acquired by a body falling freely from the directrix to that point.

If a body be projected with a given velocity so as to pass through a given point, construct the direction of projection.

14. A heavy particle slides down an inclined plane of given height under the action of gravity; find the time of descent and the velocity acquired.

If at the bottom of the inclined plane it rebound from a hard horizontal plane, what must be the inclination of the former that the range on the latter may be the greatest possible?

15. How may a pendulum be made to oscillate in a cycloid?

A pendulum which oscillates seconds at one place is carried to a place where it gains two minutes a day; compare the force of gravity at the latter place with that at the former.

FRIDAY, *Jan.* 5. $1\frac{1}{2}...4$.

1. Distinguish between compressible and incompressible fluids. Explain how fluid pressure is measured.

In the equation $p = k\rho$, which connects the pressure and density of an elastic fluid, k is a quantity of two dimensions with respect to space; shew why k is a quantity of this kind.

2. The surface of a heavy incompressible fluid at rest is a horizontal plane.

Is this true practically of the surface of a liquid contained in a vessel of finite dimensions?

3. Define specific gravity.

The specific gravity of coal is about 1.12, that of water being 1, and a cubic foot of water weighs 1000 oz.; find the edge of a cubical block of coal which weighs 2000 tons.

4. Determine the whole pressure on a surface immersed in a heavy fluid of uniform density.

What must be the vertical angle of a conical vessel, in order that when it is placed with its vertex upwards, and filled with heavy fluid through a hole at the vertex, the pressure on the curved surface may be to the pressure on the base as 4 to 3?

Prove that the ratio above-mentioned cannot for any cone be less than 2 : 3.

5. Describe Nicholson's Hydrometer, and shew how it may be applied to compare the specific gravities of two fluids.

6. If the atmosphere be supposed to be divided into indefinitely thin strata of equal thickness, the density of the air in those strata will be in geometrical progression.

7. Describe Smeaton's Air-pump, and find the density of the air in the receiver after any number of ascents of the piston.

If instead of the receiver we use a cylindrical vessel of ten times the capacity of the barrel, and cover the upper extremity with a diaphragm capable of sustaining only half the pressure of the atmosphere, find after how many ascents of the piston the diaphragm will burst.

Given $\log_{10} 2 = 0.3010300,$

$\log_{10} 11 = 1.0413927.$

8. Determine the conditions of equilibrium of a floating body.

A cylindrical vessel, the radius of the base of which is 1 foot, contains water; if a cubic foot of cork (sp. gr. = .24) be allowed to float in the water, find the additional pressure sustained by the curved surface, and by the base, respectively.

9. Explain the formation of dew. Why is dew so much more copious in hot than in cold weather; and why is the appearance of abundance of dew in the morning an indication that the day will be fine?

10. Enunciate the laws of reflexion and refraction; and state what you consider the most searching test of the truth of the latter.

11. When rays diverging from a point are incident on a plane mirror, prove that the reflected rays diverge accurately from a point.

Within what space must the eye be situated to see a given point by reflexion at the mirror; and within what space must a point be situated to be seen by the eye in a given position?

12. Find the geometrical focus of a pencil of parallel rays reflected at a spherical mirror; and prove that the intersection of any ray with the axis moves in the direction of the incident light, or in the contrary direction, (according as the mirror is concave or convex,) as the ray considered moves from the axis.

13. When diverging rays are incident nearly perpendicularly upon a spherical refracting surface, the distance of the focus of incident rays from the principal focus of rays coming in a contrary direction, is to its distance from the centre of the refractor, as its distance from the surface to its distance from the geometrical focus of refracted rays.

If the conjugate foci are each at a distance from the surface equal to twice the radius, what is the index of refraction?

14. Determine by a geometrical construction the principal focus of a lens of inconsiderable thickness.

15. Describe the human eye, and the defects of long sight and short sight; and shew how they may be remedied by the use of spectacles.

16. Draw a figure representing the course of an oblique pencil through Gregory's telescope, explaining the principal parts of the figure.

SATURDAY, *Jan.* 6. 9...12.

1. Enunciate and prove Newton's fourth Lemma.

In Lemma X., if the velocity vary as the square of the time, shew that the space will vary as the cube of the time.

2. Define similar curves, and shew that all parabolas are similar to each other.

Describe an instrument which is adapted for drawing curves similar to given curves.

3. Enunciate and prove Lemma IX.

4. A body which moves in any plane curve, and, a radius being drawn to a point either fixed or moving uniformly in a right line, describes areas about that point proportional to the times, is acted on by a centripetal force tending to that point.

5. A body describes a parabola under the action of a force parallel to the axis; determine the law of force.

Find the velocity at any point, and the time of moving from the vertex to the extremity of the latus rectum.

6. Find the law of force under the action of which a body may describe an ellipse, one of the foci being the centre of force.

If v, v' be the velocities at the extremities of any focal chord, and u that at the extremity of the latus rectum, then will v^2, u^2, v'^2 be in arithmetical progression.

7. A body moves in the circumference of a circle; find the law of force tending to an external point. Is the force attractive or repulsive?

8. Given the velocity at any points whatever of an orbit described by a body under the action of a central force, find the position of the centre.

9. Describe the change of appearance presented by the starry heavens in the course of a night. How would the phenomena be altered to a person travelling southwards; and how would the notion he might have previously formed that the Earth was isolated in space be confirmed by his journey? How does it appear that the Earth must be regarded as a point with respect to the distances of the stars?

10. Describe the Transit Instrument, without entering into an account of its adjustments.

11. Define the terms Meridian, Vertical Line, Zenith. How do we arrive at the conclusion, that the vertical line

is not the line joining the place considered with the centre of the Earth; and what are the physical causes why the two are different?

12. How does it appear from observation that the apparent path of the Sun among the fixed stars is a great circle? Why are the points of intersection of the Ecliptic and Equator called Equinoxes?

Describe the diurnal and annual motions of the Sun as they would appear to a person at the North Pole?

13. Define sidereal time, solar time, and mean solar time.

How are differences of terrestrial longitude determined by means of chronometers?

14. What is meant by the Equation of Time, and to what two causes is it principally due? How do the parts depending on these two causes respectively alter in the course of a year?

15. Define parallax. By what sort of observations are the parallaxes of the Moon and Sun respectively determined? Why are transits of Venus more valuable for the latter purpose than transits of Mercury?

16. Explain how the attractions of the Sun and Moon produce the tides; and account for spring tides and neap tides.

SATURDAY, *Jan.* 6. 1...4.

1. Through a point C in the circumference of a circle, two straight lines ACB, DCE, are drawn, cutting the circle in B and E; prove that the straight line which bisects the angles ACE, DCB meets the circle in a point equidistant from B and E.

2. Two circles intersect in A and B. At A, the tangents AC, AD are drawn to each circle and terminated by

the circumference of the other. If BC, BD be joined, shew that AB, or AB produced if necessary, bisects the angle CBD.

3. Draw a line to touch one given circle, so that the part of it contained by another given circle shall be equal to a given straight line not greater than the diameter of this latter circle.

4. Reduce to its simplest form the expression

$$\frac{(1-a^2)(1-b^2)(1-c^2)-(c+ab)(b+ca)(a+bc)}{1-a^2-b^2-c^2-2abc}.$$

5. Find a whole number which is greater than three times the integral part of its square root by unity. Shew that there are two solutions of the problem and no more.

6. If $\phi - a$, ϕ, $\phi + a$ be three angles whose cosines are in harmonical progression, shew that

$$\cos\phi = \sqrt{2}\,.\cos\frac{a}{2}.$$

7. A person, wishing to ascertain his distance from an inaccessible object, finds three points in the horizontal plane at which the angular elevation of the summit of the object is the same. Shew how the distance may be found.

8. Draw a parabola to touch a given circle in a given point, so that its axis may touch the same circle in another given point.

9. If a circle be described touching the axis major of an ellipse in one of the foci, and passing through one extremity of the axis minor, the semi-axis major will be a mean proportional between the diameter of the circle and the semi-axis minor.

10. If AB, CD, two lines in an ellipse, not parallel to each other, make equal angles with either axis; the lines AC, BD and AD, BC will also make equal angles with either axis.

11. Two forces F and F', acting in the diagonals of a parallelogram, keep it at rest in such a position that one of its edges is horizontal; shew that $F \sec \alpha = F' \sec \alpha' = W \operatorname{cosec} (\alpha + \alpha')$, where W is the weight of the parallelogram, α and α' the angles between its diagonals and the horizontal side.

12. A quadrilateral figure possesses the following property; any point being taken and four triangles formed by joining this point with the angular points of the figure, the centres of gravity of these triangles lie in the circumference of a circle: prove that the diagonals of the quadrilateral are at right angles to each other.

13. If the angle of a hollow cone polished internally be any submultiple of 180°, a cylindrical pencil of rays incident parallel to the axis will after a certain number of reflexions be a cylindrical pencil parallel to the axis, and of the same diameter as the incident pencil.

14. A cubical box is half filled with water and placed upon a rough rectangular board, so as to have the edges of its base parallel to those of the rectangle; if the board be slowly inclined to the horizon, determine whether the box will slide down or topple over.

15. A body floats in a mixture of two given fluids with a volume A immersed; one-half of the mixture being removed, and its place supplied by an equal quantity of the lighter fluid, the same body floats with a volume $A + B$ immersed. Determine the ratio of the quantities of fluid in the original mixture, supposing the volume of the mixture to be equal to the sum of the volumes of the component fluids.

Explain the result when the densities of the fluids are as $A + B$ to $A - B$.

16. There are two walls of equal known height at right angles to each other, and running in known directions; shew how to find the Sun's altitude and azimuth by observing the breadth of the shadows of the two walls

at any given time. And prove that the sum of the squares of the breadths of the shadows will be the same whatever be the direction of the walls.

17. If the same two stars rise together at two places, the places will have the same latitude. And if they rise together at one place and set together at the other, the places will have equal latitudes, but one North and the other South.

18. A body is projected from a given point in a horizontal direction with given velocity, and moves upon an inclined plane passing through the point. If the inclination of the plane vary, find the locus of the directrix of the parabola which the body describes.

19. An imperfectly elastic ball lies on a billiard-table, determine the direction in which an equal ball must strike it in order that they may impinge upon a side of the table at equal given angles.

20. The circle described through two points of an equiangular spiral and the point of intersection of the tangents at those points will pass through the pole. Prove this, and apply the proposition to shew that the curvature at any point of an equiangular spiral varies inversely as the distance of the point from the pole.

21. A bead running upon a fine thread the extremities of which are fixed describes an ellipse in a plane passing through the extremities, under the action of no external force; prove that the tension of the thread for any given position of the bead is inversely proportional to the square of the conjugate diameter.

22. The centres of two equal spheres (elasticity e, radius r,) move in opposite directions in a circle (radius R) about a centre of force varying inversely as the square of the distance; determine the motion of the spheres after they have impinged, supposing that $e = \dfrac{r^2}{R^2 - r^2}$; and prove that the latus rectum of the conic section described after the second impact will be $2e^2 R$.

1850.

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THURSDAY, *Jan.* 3. 9...12.

1. THE opposite sides and angles of parallelograms are equal to one another, and the diameter bisects them, that is, divides them into two equal parts.

If the opposite sides, or the opposite angles of any quadrilateral figure be equal, or if its diagonals bisect one another, the quadrilateral is a parallelogram.

2. Describe a square which shall be equal to a given rectangle.

Given a square and one side of a rectangle which is equal to the square, find the other side.

3. In a circle, the angle in a semicircle is a right angle; the angle in a segment greater than a semicircle is less than a right angle; and the angle in a segment less than a semicircle is greater than a right angle.

The greatest rectangle that can be inscribed in a circle is a square.

4. Cut off a segment from a given circle which shall contain an angle equal to a given rectilineal angle.

Divide a circle into two segments such that the angle in one of them shall be five times the angle in the other.

5. Describe an isosceles triangle, having each of the angles at the base double of the third angle.

Shew that the base of the triangle is equal to the side of a regular pentagon inscribed in the smaller circle of the figure.

6. Find a third proportional to two given straight lines.

AB is a diameter, and P any point in the circumference of a circle; AP and BP are joined and produced if necessary: if from any point C of AB a perpendicular be drawn to AB meeting AP and BP in points D and E respectively, and the circumference of the circle in a point F, shew that CD is a third proportional to CE and CF.

7. If two straight lines be at right angles to the same plane, they shall be parallel to one another.

8. If a line be drawn bisecting the angle between the focal distance of any point of a parabola and a perpendicular from that point upon the directrix, every point of the line will lie outside the parabola.

9. The perpendiculars from the foci on the tangent of an ellipse intersect the tangent in the circumference of a circle having the axis major as diameter.

Employ this proposition to find the locus of the intersection of a pair of tangents at right angles to each other.

10. The rectangle under the abscissæ of any diameter of an ellipse is to the square of the semiordinate, as the square of the diameter is to the square of the conjugate.

$$(PV \,.\, VG : QV^2 :: CP^2 : CD^2).$$

11. In the hyperbola the rectangle under the lines intercepted between the centre and the intersections of the axis with the ordinate and tangent respectively is equal to the square of the semiaxis major.

$$(CN \,.\, CT = AC^2).$$

Through N draw NQ parallel to AP to meet CP in Q; prove that AQ is parallel to the tangent at P.

12. If a point K be taken in the major axis of the hyperbola such that CK is a third proportional to CS and CA, and a perpendicular to the axis be drawn through K, the distance of any point in the curve from this line will bear a constant ratio to its distance from the point S.

13. If a right cone be cut by a plane, which is parallel to a line in the surface, and perpendicular to the plane containing that line and the axis, the section is a parabola.

What is the section when the cutting plane is parallel to a generating line but not perpendicular to the plane containing the axis and that line?

14. Find the diameter of curvature at any point of an ellipse.

If an ellipse, a parabola, and a hyperbola have a common tangent and the same curvature at the vertex, the ellipse will lie entirely within the parabola and the parabola entirely within the hyperbola.

THURSDAY, *Jan.* 3. $1\frac{1}{2}...4$.

1. EXPLAIN how numbers are represented in the decimal notation.

From the number represented by 5063 subtract that represented by 3297, explaining every part of the process; and thence infer the general rule.

2. Shew how to find the least whole number which is accurately divisible by each of two given whole numbers.

Find the least number of ounces of standard gold that can be coined into an exact number of half-sovereigns; standard gold being coined at the rate of £3. 17*s*. $10\frac{1}{2}$*d*. to an ounce.

3. If a quantity vary directly as a when b is invariable, and inversely as b when a is invariable; prove that it will vary as $\frac{a}{b}$ when both a and b are variable.

If 5 men and 7 boys can reap a field of corn of 125 acres in 15 days; in how many days will 10 men and 3 boys reap a field of corn of 75 acres, each boy's work being $\frac{1}{3}$ of a man's?

4. If four quantities be proportionals according to Euclid's definition, shew that they will be proportionals according to the algebraic definition.

If $a : a' = b : b' = c : c' =$ &c. shew that

$$a : a' = a\alpha + b\beta + c\gamma + \dots : a'\alpha + b'\beta + c'\gamma + \dots$$

$\alpha, \beta, \gamma \dots$ being any quantities.

5. A mixture of soda and potass, dissolved in 2540 grains of water, took up 980 grains of aqueous sulphuric acid, and the weight of the compound solution was 4285 grains. Find how much potass, and how much soda the mixture contained, assuming that aqueous sulphuric acid unites with soda in the proportion of 49 grains to 32, and with potass in the proportion of 49 to 48.

6. Solve the equations:

(1) $$m\left(\frac{x+a}{x+b}\right) + n\left(\frac{x+b}{x+a}\right) = m + n.$$

(2) $$x^2 + 2ax + b^2 = 0.$$

(3) $$\begin{cases} a(y+z) - yz = 0, \\ b(z+x) - zx = 0, \\ c(x+y) - xy = 0. \end{cases}$$

7. Find the sum of a series of quantities in geometrical progression, and apply the result to find a common fraction equivalent to a recurring decimal fraction.

If a be the first, and l the last of a series of n quantities in geometrical progression, prove that the continued product of the terms of the series is $(al)^{\frac{n}{2}}$.

8. Define the tangent of an angle, and trace its changes of sign and magnitude through the four quadrants.

Find when $\sin A$ and $\text{vers}\, A$ will have the same sign.

9. Prove the following formulæ:

(a) $\sin(A + B) = \sin A \cos B + \sin B \cos A$

(b) $\sin 2A = 2 \sin A \cos A$.

(c) $\cos 3A = 4\cos^3 A - 3\cos A$.

Find $\cos 30^\circ$, and $\sin 63^\circ$.

10. Shew that in general it will be possible to determine two triangles in which two sides and the angle opposite to the less are of given magnitudes.

If a, b, B be given, and a be $> b$, and if c, c' be the two values found for the third side of the triangle, then

$$c^2 - 2cc' \cos 2B + c'^2 = 4b^2 \cos^2 B.$$

11. If a, b, c be the sides of a triangle, prove that its area is equal to

$$\tfrac{1}{4}\sqrt{2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4}.$$

Apply this expression to find the area when the angle opposite to c is a right angle.

12. If $A + B + C = 180^0$, prove that $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.

If α, β, γ denote the distances from the angular points of a triangle to the points of contact of the inscribed circle, shew that the radius of the circle is equal to

$$\left(\frac{\alpha\beta\gamma}{\alpha + \beta + \gamma}\right)^{\frac{1}{2}}.$$

13. A tower 51 feet high has a mark at the height of 25 feet from the ground; find at what distance the two parts subtend equal angles to an eye at the height of 5 feet from the ground.

14. Define a logarithm, and shew how to deduce the logarithm of a number to base b from its logarithm to base a.

FRIDAY, *Jan.* 4. 9...12.

1. ASSUMING that the diagonal of a parallelogram represents the direction of the resultant of two forces acting at a point, the magnitudes and directions of the forces being represented by the sides of the parallelogram, shew that the diagonal will also represent the magnitude of the resultant.

2. If three forces which act in a plane keep a rigid body at rest; prove that their lines of action either are parallel, or pass through a point, and in both cases shew that any two of the forces are inversely proportional to the perpendiculars drawn on their respective lines of action from any point in the line of action of the third.

An uniform heavy rod of given length is to be supported in a given position, with its upper end resting at a given point against a smooth vertical wall, by means of a fine thread attached to the lower end of the rod and to a point in the wall. Find by a geometrical construction the point in the wall to which the string must be attached.

3. Find the ratio of the power to the weight necessary for equilibrium on the wheel and axle.

If the axis about which the machine turns, coincide with that of the axle, but not with the axis of the wheel, find the greatest and least ratios of the power and weight necessary for equilibrium, neglecting the weight of the machine.

4. Define the centre of gravity of a heavy body, and prove that a body can have but one centre of gravity.

Find the centre of gravity of a plane triangle.

5. Five pieces of an uniform chain are hung at equidistant points along a rigid rod without weight, and their lower ends are in a straight line passing through one end of the rod; find the centre of gravity of the system.

Also shew that if the system balance about a point of the rod in one position, it will balance about it in any position.

6. Find the ratio of the power to the weight necessary for equilibrium on an inclined plane, when the power acts along the plane.

If the inclined plane be the upper surface of a wedge whose under surface rests on a smooth horizontal table, find the horizontal force which must act on the wedge to keep it at rest.

7. In the system of pulleys in which the strings are parallel and each pulley hangs by a separate string, find the ratio of the power to the weight necessary for equilibrium on the principle that any motion of P along its line of action : resulting motion of W :: W : P.

8. Explain clearly on what conventions with respect to units the equation $P = Mf$ is true, where f expresses the accelerating effect of a force whose statical measure is P.

A body weighing 10lbs. is moved by a constant force which generates in a second a velocity of 1 foot per second; find what weight the force would support.

9. Shew that in uniformly accelerated motion

$$s = \frac{1}{2}ft^2.$$

A body falling in vacuo, under the action of gravity, is observed to fall through 144.9 feet and 177.1 feet in two successive seconds; determine the accelerating force of gravity, and the time from the beginning of the motion.

10. Given the velocities of two bodies of which the masses are M, M' and the elasticity e; find the velocity of each body after a direct collision.

Three equal balls are moving in the same direction with velocities which are proportional to 3, 2, 1, and the distances between them were at a given time the same; shew that after impact the velocities will continue to be in arithmetical progression.

11. Prove that if a heavy body fall down a smooth curve, the velocity at any point will be that due to the vertical height through which it has fallen.

Shew how to place a plane of given length in order that a body may acquire a given velocity by falling down it.

12. Find the curve described by a body projected in vacuo with a given velocity and in a given direction, explaining the application of the second law of motion to the problem.

A smooth tube of uniform bore is bent into the form of a circular arc greater than a semicircle and placed in a vertical plane with its open ends upwards and in the same horizontal line. Find the velocity with which a ball that fits the tube must be projected along the interior from the lowest point, in order that it may pass out at one end and re-enter at the other.

13. If a particle oscillate in a cycloid, the time of an oscillation will be independent of the arc of vibration.

A seconds pendulum was too long on a given day by a small quantity a, it was then over-corrected so as to be too short by a during the next day: shew that the number of minutes gained in the two days was $1080\frac{a^2}{L^2}$ nearly, if L be the length of the seconds pendulum.

FRIDAY, *Jan.* 4. $1\frac{1}{2}$...4.

1. FIND the pressure referred to a unit of area at any depth below the surface of a heavy incompressible fluid.

If from every point in the vertical side of a rectangular vessel containing fluid a horizontal line be drawn proportional to the pressure at that point; find the locus of the extremities of such lines; and thence deduce the amount of the whole pressure upon one of the vertical sides of a cube filled with fluid, and the point of application of the resultant of the pressures.

2. Determine the conditions of equilibrium of a floating body.

3. Define specific gravity.

Given weights of substances of known specific gravity are compounded; find the specific gravity of the compound.

Eleven ounces of gold (sp. gr. 19.3) are mixed with one

ounce of copper (sp. gr. 8.8), find the specific gravity of the compound, supposing its volume to be the sum of the volumes of the two metals.

4. Describe the Barometer.

Shew how the common barometer, which consists of a tube plunged into a cylindrical vessel, may be graduated so that the graduation may give correctly the pressure of the atmosphere.

5. Describe Nicholson's hydrometer.

Given two weights which cause the instrument to sink to a certain depth in two fluids, find the weight which will make it sink to the same depth in a mixture of known volumes of the two fluids.

6. Give the experiment from which it is inferred that the pressure of air at a given temperature varies inversely as the space it occupies.

If the temperature vary, what relation exists between the pressure, the volume, and the temperature?

A given quantity of air under the pressure of m pounds to the square inch occupies n cubic inches when the temperature is t, find how many cubic inches it will occupy under the pressure of m' pounds to the square inch when the temperature is t'.

7. Shew how to graduate a thermometer, and to compare the scale of two differently graduated thermometers.

The number which expresses a certain temperature on the centigrade scale is equal to the sum of the numbers which express the same temperature on Fahrenheit's and Reaumur's respectively; find the numbers.

8. A luminous point is placed between two plane mirrors inclined to each other at a given angle; find the position of the images, and shew that their number is limited.

9. Diverging rays are incident directly upon a concave refracting surface; find the position of the geometrical focus.

Trace the corresponding positions of the conjugate

foci, and find where the distance between them is the greatest, when they are between the centre and the surface.

10. A ray of light is refracted through a prism in a plane perpendicular to its edge; find the deviation produced by the refraction.

A speck is situated just within a glass sphere; shew how much of the surface of the sphere must be covered in order that the speck may be invisible at all points outside the sphere on a line drawn from the speck through the centre.

11. Point out the distinction between a real and a virtual image.

A plane mirror is placed perpendicular to the axis of a concave spherical reflector, nearer to the principal focus than to the face, and between them; rays from a very distant object fall directly upon the spherical mirror; trace the pencil by which an eye will see the image formed after two reflexions at each mirror.

12. When an object is viewed through a convex lens, shew that there are circumstances under which the image may be erect or inverted, magnified or diminished.

13. What advantages are attained by the use of a telescope, and how are these advantages limited in practice? Describe the common astronomical telescope.

14. Trace a pencil of rays through Newton's telescope.

Shew that the magnifying power is equal to the ratio of the diameter of a direct pencil which fills the object mirror to the diameter of the pencil which emerges from the eye-glass.

SATURDAY, *Jan.* 5. 9...12.

1. ENUNCIATE and prove Newton's second Lemma.

Hence shew that two quantities may vanish in an infinite ratio to each other.

2. State and prove the seventh Lemma.

Define a subtense, and explain the bearing of your definition upon the proof of the Lemma.

3. If a body move in free space under the action of a central force, the velocity at any point of the orbit varies inversely as the perpendicular let fall from the centre upon the tangent.

If lines, proportional to the Earth's velocity, and always parallel to the direction of its motion, be drawn from a fixed point, shew that the extremities of these lines will trace out a circle.

4. Having given the velocities at any points whatever of a curve which a body describes under the action of forces tending to a common centre; find that centre.

5. A body describes an ellipse, find the law of the centripetal force which tends to the centre of the ellipse.

6. If several bodies revolve about a common centre, and the centripetal force vary inversely as the square of the distance from the centre, the latera recta of the orbits are to each other in the duplicate ratio of the areas which the radii drawn from the bodies to the centre describe in the same time.

7. Given that the centripetal force varies inversely as the square of the distance, and that the absolute force is known; determine the curve which a body describes after being projected from a given point with a given velocity and in a given direction.

The problem is to be solved for one case only.

8. Describe, in their chief features, the apparent motions of the fixed stars and of the Sun; and, supposing these appearances to arise solely from the motion of the Earth, deduce the nature of the Earth's motion.

9. Define the equator and ecliptic; and explain how the position of a heavenly body is determined (1) by declination and right ascension, (2) by latitude and longitude.

10. Describe accurately the errors of adjustment to which a transit instrument is liable; and, in the case of each error, mention the positions of the heavenly bodies whose observed times of transit are least (or not at all) affected by it.

11. Explain how the meridian altitude of a heavenly body is determined by a double observation with the mural circle.

When the meridian altitude of a heavenly body is known, what other elements are required in order to determine its position among the fixed stars?

12. Account for the changes in the Moon's phases.

Shew that the full Moon remains above the horizon of a place during nearly the whole of the night.

Explain the cause of a lunar eclipse.

13. Define sidereal, solar, and mean solar time. Define also the equation of time.

If the Earth's orbit were circular, find when the equation of time would change sign from positive to negative, and when from negative to positive.

14. Explain the nature of aberration, and its effect on the apparent positions of the fixed stars.

Shew that, at any given time, all stars which lie in a certain great circle have no aberration in right ascension. Also give a geometrical construction for finding stars that have no aberration in declination at a given time; and prove that the locus of such stars becomes a great circle twice a year.

SATURDAY, *Jan.* 5. 1...4.

1. A NUMBER of persons were engaged to do a piece of work which would have occupied them m hours if they had commenced at the same time, but instead of doing so they commenced at equal intervals and then continued to work till the whole was finished: the payment being proportional to the work done by each, the first comer received r times as much as the last; find the time occupied.

2. If $ABCD$ be a parallelogram, and P, Q two points in a line parallel to AB, and if PA, QB meet in R and PD, QC in S, prove that RS is parallel to AD.

3. Two sides of a triangle whose perimeter is constant are given in position; shew that the third side always touches a certain circle.

4. Shew that the product of the terms of an arithmetical progression is greater than $(al)^{\frac{n}{2}}$; and that the sum of the terms of a geometrical progression is less than $(a+l)\frac{n}{2}$; where in both cases a, l and n denote the first and last terms and the number of terms respectively.

5. If from any point P of a circle, PC be drawn to the centre C, and a chord PQ be drawn parallel to the diameter ACB and bisected in R, shew that the locus of the intersection of CP and AR is a parabola.

6. From P a point in an ellipse lines are drawn to A, B the extremities of the major axis, and from A, B lines are drawn perpendicular to AP, BP; shew that the locus of their intersection will be another ellipse, and find its axes.

7. If two ellipses, having the same major axes, can be inscribed in a parallelogram, the foci of the ellipses will lie in the corners of an equiangular parallelogram.

8. If from the extremities of any diameter of an equilateral hyperbola lines be drawn to any point in the curve, they will be equally inclined to the asymptotes.

9. A person wishing to ascertain the distances between three inaccessible objects A, B, C places himself in a line with A and B; he then measures the distances along which he must walk in a direction at right angles to AB, until A, C and B, C respectively are in a line with him, and also observes in those positions their angular bearings; shew how he can find the distances between A, B, C.

10. A heavy body is supported in a given position by means of a string which is fastened to two given points in the body, and then passes over a smooth peg; find the length of the string.

11. Two spheres are supported by strings attached to a given point and rest against one another; find the tensions of the strings.

12. Shew that it is possible to project a ball on a smooth billiard-table from a given point in an infinite number of directions so as after striking all the sides in order once or oftener to hit another given point; but that this number is limited if it have to return to the point from which it was projected.

13. A cone of given weight W is placed with its base on a smooth inclined plane and supported by a weight W' which hangs by a string fastened to the vertex of the cone and passing over a pulley in the inclined plane at the same height as the vertex. Find the angle of the cone when the ratio of the weights is such that a small increase of W' would cause the cone to turn about the highest point of the base, as well as slide.

14. A conical vessel containing a given quantity of fluid has its axis vertical, and another cone with the same vertical angle is placed to float in the fluid with its vertex

downwards; find how much the fluid will rise in consequence.

15. A hollow cylinder containing air is fitted with an air-tight piston which when the cylinder is placed vertically is at a given height above the base; the cylinder being now inverted and placed vertically in a fluid sinks partly below the surface; find the position of equilibrium.

16. If a luminous point be seen after reflexion at a plane mirror by an eye in a given position, there is a certain space within which the image of the point can never be situated, however the position of the plane mirror be changed; find this space.

17. If α be the angle which every diameter of a circular disc subtends at a luminous point, shew that the ratio of the light which falls on the disc to the whole light emitted is as $\sin^2 \frac{\alpha}{4} : 1$.

18. If any number of particles be moving in an ellipse about a force in the centre, and the force suddenly cease to act, shew that after the lapse of $\left(\frac{1}{2\pi}\right)^{\text{th}}$ part of the period of a complete revolution all the particles will be in a similar, concentric and similarly situated ellipse.

19. Prove that all stars which rise at the same instant at a place within certain limits of latitude will, after a certain interval, lie in a vertical great circle; and determine those limits.

20. Shew how to find the days of the year on which the light of the sun reflected by a given window which has a south aspect will be thrown into some one of the lower windows of an opposite range of buildings.

21. Two perfectly elastic balls are moving in concentric circular tubes in opposite directions and with velocities proportional to the radii: at an instant when they

are in the same diameter and on opposite sides of the centre the tubes are removed and the balls move in ellipses under the action of a force of attraction in the common centre of the circles varying inversely as the square of the distance. After one has performed in its orbit a complete revolution and the other a revolution and a half, a direct collision takes place between the balls and they interchange orbits; find the relation between the radii of the circles and between the masses of the balls.

1851.

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TUESDAY. *Dec.* 31. 9...12.

1. TRIANGLES upon equal bases and between the same parallels are equal to one another.

Let ABC, ABD be two equal triangles upon the same base AB and on opposite sides of it; join CD meeting AB in E; shew that CE is equal to ED.

2. In any right-angled triangle, the square described upon the side subtending the right angle, is equal to the squares described upon the sides which contain the right angle.

If ABC be a triangle whose angle A is a right angle, and BE, CF be drawn bisecting the opposite sides respectively; shew that four times the sum of the squares of BE and CF is equal to five times the square of BC.

3. The opposite angles of any quadrilateral figure inscribed in a circle are together equal to two right angles.

If a polygon of an even number of sides be inscribed in a circle, the sum of the alternate angles together with two right angles is equal to as many right angles as the figure has sides.

4. Inscribe an equilateral and equiangular quindecagon in a given circle.

In a given circle inscribe a triangle whose angles are as the numbers 2, 5 and 8.

5. If the angle of a triangle be divided into two

equal angles by a straight line which also cuts the base, the segments of the base have the same ratio which the other sides of the triangle have to one another.

If A, B, C be three points in a straight line, and D a point at which AB and BC subtend equal angles, shew that the locus of the point D is a circle.

6. If two straight lines be parallel, and one of them be at right angles to a plane, the other is at right angles to the same plane.

From a point E draw EC, ED perpendicular to two planes CAB, DAB which intersect in AB, and from D draw DF perpendicular to the plane CAB meeting it in F; shew that the line joining the points C and F, produced if necessary, is perpendicular to AB.

7. In the parabola if from the extremity Q of an ordinate QV, a perpendicular QD be let fall on the diameter PV, then $QD^2 = 4AS.PV$.

8. Assuming that the sum of the focal distances of a point in the ellipse is equal to a given line, shew that the axis major is equal to the same line.

Shew that the axis major is greater than any other diameter.

9. In the ellipse if PU be a tangent at P meeting the minor axis produced in U, and PN be drawn perpendicular to the minor axis, then

$$CN : CB :: CB : CU.$$

If a series of ellipses be described having the same major axis, the tangents at the extremities of their latera recta will all meet the minor axis in the same point.

10. Define conjugate diameters, and prove that if the semidiameter CD be conjugate to CP, then CP will be conjugate to CD.

11. In the hyperbola the rectangle under the abscissæ of any diameter is to the square of the ordinate, as the square of the diameter to the square of the conjugate.

12. In the parabola, at any point P, the chord of curvature parallel to the axis and that through the focus are severally equal to $4SP$.

If the circle of curvature at the point P intersect the parabola in another point Q, and QR, drawn parallel to the axis, meet the circle in R, shew that PR is the chord of curvature through the focus.

13. If a right cone be cut by a plane which meets the cone on both sides of the vertex, the section is a hyperbola.

Shew how to cut from a given cone a hyperbola whose asymptotes shall contain the greatest possible angle.

TUESDAY, *Dec.* 31. $1\frac{1}{2}$...4.

1. How many Roubles at 3*s.* $4\frac{1}{2}$*d.* to the Rouble are equal in value to 378 Napoleons at 15*s.* $9\frac{3}{4}$*d.* to the Napoleon?

2. Prove the rule for finding the greatest common measure of two numbers.

Shew that the greatest common measure of the two numbers is equal to the greatest common measure of any divisor made use of in the process and the corresponding dividend.

3. Find the square root of 3 to five places of decimals, and determine (without multiplication) the square of the approximate root.

4. Shew that $(x+y)^7 - x^7 - y^7$ is exactly divisible by $(x^2+xy+y^2)^2$.

5. Solve the equations

(1) $$\frac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}} + \frac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}} = 34.$$

(2) $$(x-a)(x-b)(x-c)+abc = 0.$$

(3) $$x^2 + yz = y^2 + zx = c,\quad z^2 + xy = a.$$

6. Find the sum of a series of quantities in arithmetical progression.

The square of the arithmetic mean of two quantities is equal to the arithmetic mean of the arithmetic and geometric means of the squares of the same two quantities.

7. Find the number of combinations of n things taken r together.

Shew that for a given even value of n the number of combinations is greatest when $r = \frac{1}{2}n$.

8. Compare the magnitudes of two angles which contain the same number of French and English degrees respectively.

Divide an angle which contains n degrees into two parts, one of which contains as many English minutes as the other does French.

9. Express the values of the tangent, secant and cosecant in terms of the sine of the angle and also in terms of the cosine of twice the angle.

Find the values of the tangent, secant, and cosecant, of $22^\circ\ 30'$.

10. Prove that $\sin (A + B) = \sin A \cos B + \cos A \sin B$, and deduce a similar expression for $\cos (A + B)$.

If $a \tan A + b \tan B = (a + b) \tan \dfrac{A + B}{2}$, shew that $\dfrac{a}{b} = \dfrac{\cos A}{\cos B}$.

11. In a right-angled triangle in which C is the right angle, prove that $\cot \dfrac{A}{2} = \dfrac{b + c}{a}$.

Shew also that in any triangle

$$(b + c - a) \tan \frac{A}{2} = (c + a - b) \tan \frac{B}{2} = (a + b - c) \tan \frac{C}{2},$$

and hence that each of these quantities is equal to

$$\left\{\frac{(b+c-a)(c+a-b)(a+b-c)}{a+b+c}\right\}^{\frac{1}{2}}.$$

12. Find the area of a triangle (1) in terms of two sides and the angle between them, (2) in terms of two angles and the side between them.

13. Define the characteristic and mantissa of a logarithm; and find the characteristic of the logarithm of 5 when the base is 3, and of the logarithm of $\frac{1}{3}$ when the base is 5.

14. If θ be an angle determined from the equation $\cos\theta = \dfrac{a-b}{c}$; prove that in any triangle

$$\cos\frac{A-B}{2} = \frac{(a+b)\sin\theta}{2\sqrt{ab}} \text{ and } \cos\frac{A+B}{2} = \frac{c\sin\theta}{2\sqrt{ab}}.$$

WEDNESDAY, *Jan.* 1. 9...12.

1. THE sum of the moments of any two forces is equal to the moment of their resultant, whatever be the point about which the moments are taken.

2. Find the condition of equilibrium on the inclined plane, the power acting parallel to the base.

3. Find the relation between the power and weight in the system of pullies in which each string is attached to the weight, taking into account the weight of the pullies.

4. If a point O be kept in equilibrium by forces represented in magnitude and direction by the lines OP, OQ, OR, the point O is the centre of gravity of the triangle PQR.

5. If two weights support each other on inclined planes by means of a string passing over the common vertex of the two planes, and the system is set in motion, the centre of gravity of the weights moves in a horizontal line.

6. Two uniform heavy rods equal in length and weight, connected at their ends by a hinge are placed astride across a smooth horizontal cylinder, determine the position of equilibrium and also the tension of the hinge.

7. What is meant by the equilibrium of a system being 'stable' or 'unstable?' In what case is the equilibrium of two weights acting at the extremities of a bent lever, stable?

8. Write down the laws of motion; giving any illustrations you please for the sake of explanation.

If a weight of ten pounds be placed upon a plane which is made to descend with a uniform acceleration of 10 feet per second, what is the pressure upon the plane?

9. If a body move from rest under the action of a uniform accelerating force, prove that the space moved over varies as the square of the time of motion.

If a body fall down an inclined plane, and another be projected from the starting point horizontally along the plane; find the distance between the two bodies when the first has descended through a given space.

10. Find the angle which the direction of a projectile makes with the horizon at any point of its path, and determine its distance from a line drawn through the point of projection parallel to this direction.

11. When a heavy particle falls down a smooth curve, the velocity at any point is that due to the vertical height through which it has fallen.

12. A ball impinges directly with a given velocity upon another ball at rest; find the velocity of each after impact, their common elasticity being e.

If the vis viva before impact be n times the vis viva after impact, find their common elasticity.

13. Define a cycloid, and prove that the arc measured from the vertex to any point is equal to twice that chord of the generating circle which touches the curve at that point. Hence deduce the radius of curvature at the

vertex, and shew that the time of oscillation in a small arc of the generating circle will be half the time of oscillation in the cycloid.

WEDNESDAY, *Jan.* 1. $1\frac{1}{2}$...4.

1. EXPLAIN the method of estimating fluid pressure.

Shew that the pressure of a fluid will be the same at all points in the same horizontal plane, whatever be the form of the vessel containing it.

2. Find the pressure on a plane immersed in a fluid.

Distinguish between the whole pressure and the resultant pressure on a body immersed in a fluid; shew that if the body be depressed in the fluid, the former will be increased, but the latter will remain unaltered.

3. If a cone float in a fluid with its axis vertical, first with the vertex immersed, and next with the base immersed, compare the depression of the vertex below the surface of the fluid in the first case with its elevation above it in the second.

4. If a given body lose in air when the height of the barometric column is h the m^{th} part of its weight; find what part of its weight it will lose when the height of the barometric column is h', and explain how the change of atmospheric pressure will affect the apparent weight of a substance when weighed with weights of a standard substance.

5. Describe the common Barometer.

If there be a small quantity of air in the tube above the column of mercury, what will be the effect on the indications of the barometer?

A faulty barometer indicated 29.2 and 30 inches when the indications of a correct instrument were 29.4 and 30.3 respectively, find the length of tube which the air in the tube would fill under the pressure of 30 inches.

6. Describe Smeaton's Air-pump, and explain the mode of its operation.

7. A luminous point is placed between two parallel plane mirrors, find the position of the successive images.

When the luminous point moves uniformly in a straight line, shew that all the images will move uniformly in two sets of parallel straight lines which are equally inclined to the mirrors.

8. Two rays are incident at any point of a spherical mirror whose centre is E, the one parallel to the axis of the mirror, the other proceeding from a point Q in the axis, and the reflected rays cut the axis in G and q respectively; shew that $GQ.Gq = GE^2$.

If the axis AE of the spherical mirror meet the surface produced in R, shew that a ray proceeding from R and making an angle of 30^0 with the axis will be reflected to the principal focus of the mirror.

9. If parallel rays be incident directly upon a spherical refracting surface, the distance of the geometrical focus of refracted rays from the surface, is to its distance from the centre, as the index of refraction to unity.

A pencil of parallel rays is incident directly upon a spherical refracting surface, and after refraction converges to a point at a distance from the surface equal to three times the radius; find the index of refraction, (1) when the surface is concave, (2) when it is convex.

10. Find the geometrical focus of a pencil of rays after direct refraction through a lens whose thickness is inconsiderable.

Two convex lenses whose focal lengths are $3f$ and f are separated by an interval $2f$; how must a pencil of rays be incident upon the first lens, so as to emerge parallel after refraction through the second lens? Draw a figure representing the course of the pencil.

11. Shew how to find by experiment the focal length of a lens.

The least distance between an object and its image formed by a plano-convex lens of glass is 12 inches; the index of refraction being $\frac{3}{2}$, find the radius of the spherical surface.

12. Describe Galileo's telescope, and trace the course of the pencil of rays by which a given point out of the axis is seen.

THURSDAY, *Jan.* 2. 9...12.

1. ENUNCIATE and prove Newton's fourth Lemma.

Apply this Lemma to prove that the area included between a hyperbola and the tangents at the vertices of the conjugate hyperbola is equal to the area included between the conjugate hyperbola and the tangents at the vertices of the hyperbola.

2. The spaces described from rest by a body under the action of any finite force are in the beginning of the motion as the squares of the times in which they are described.

If the force vary as the time from rest, prove that the velocity will vary as the square of the time.

3. The centripetal forces of bodies which describe different circles with uniform velocities, tend to the centres of the circles, and are to each other as the squares of the arcs described in the same time divided by the radii.

Give a formula for finding the height of a body which moves under the attraction of the earth in such a way as always to be vertically above a point in the equator, stating the numerical values of the quantities appearing in the result.

4. If a body revolve about a fixed centre of force, and a sagitta be drawn to any very small arc bisecting the chord of the arc and passing through the centre of force, then the force at the middle of the arc will ulti-

mately vary directly as the sagitta and inversely as the square of the time of describing the arc.

If a body move in a curve under the action of any force, whether tending to a centre or not, the square of the velocity at any point varies as the product of the force and the chord of curvature in the direction of the force.

5. If a body move in a parabola under the action of a force parallel to the axis, prove that this force is constant: and conversely.

6. When a body revolves in an ellipse, find the law of the centripetal force tending to the focus of the ellipse.

7. If several bodies revolve round a common centre, and the centripetal force vary inversely as the square of the distance, the velocities of the bodies are in a ratio compounded of the ratio of the perpendiculars inversely, and the subduplicate ratio of the latera recta directly.

The velocity of a body revolving in any conic section is to the velocity of a body revolving in a circle at the distance of half the latus rectum as that distance is to the perpendicular from the focus upon the tangent.

8. Why does not a star always rise at the same time? Explain how the time of a star's rising and that of its setting alter in the course of the year.

Orion's belt being in the equator and having about $5^h\ 30^m$ Right Ascension; during what part of the night will it be visible at the vernal and autumnal equinoxes?

9. What must be the approximate age of the Moon that she may be seen in the south at seven o'clock in the morning at the time of the equinox? Will the convexity of the crescent appear to a spectator on his right hand or on his left?

10. For what purpose is the Mural Circle employed? Describe the instrument and its adjustments.

11. Give definitions of the ecliptic, meridian, equator and solstitial colure. Which of these great circles are

fixed circles in the celestial sphere? Which of them would, if visible, appear fixed in the sky?

12. Explain briefly the necessity or the advantage of applying corrections for Refraction, Aberration, Parallax, and Precession and Nutation to the observations of a heavenly body.

Which of these corrections do not apply to transit observations?

13. Explain fully how the longitude of a ship at sea is determined by the method of Lunar Distances.

THURSDAY, *Jan.* 2. 1...4.

1. IN AB the diameter of a circle take two points C, D equally distant from the centre, and from any point E in the circumference draw EC, ED; shew that

$$EC^2 + ED^2 = AC^2 + AD^2.$$

2. If through the fixed points P, Q parallel lines be drawn meeting two fixed parallel lines in the points M, N; then the line through the points M, N passes through a fixed point.

3. In a given circle it is required to inscribe a triangle similar and similarly situated to a given triangle.

4. Describe a circle which shall pass through two given points and cut off from a given straight line a chord of given length.

5. If $\dfrac{3x-2}{(x-1)(x-2)(x-3)}$ be expanded in a series ascending by powers of x, find the coefficient of x^n.

6. Find the sum of the different numbers which can be formed with m digits α, n digits β, &c. the entire series of $m + n +$ &c. digits being employed in the formation of each number.

7. The difference between the arithmetic and geometric means of two numbers is less than one-eighth of the squared difference of the numbers divided by the less number, but greater than one-eighth of such squared difference divided by the greater number. If x, y be any two numbers, x_1, y_1 their arithmetic and geometric means, x_2, y_2 the arithmetic and geometric means of x_1, y_1 and so on, find major and minor limits for the difference $x_n - y_n$.

8. If all the sums of two letters that can be formed with any n letters be multiplied together, then in each term of the product, the sum of any r of the indices cannot exceed the number $rn - \frac{1}{2}r(r+1)$.

9. Eliminate x from the equations

$$(x-a)(x-b) = (x-c)(x-d) = (x-e)(x-f);$$

and from the same equations, with the additional relation $e = f$, find a quadratic equation for determining the quantity e or f. Shew also that if m', m'' be the values of e or f, then $m'' - m'$ is a harmonic mean between $a - m'$, $b - m'$ and between $c - m'$, $d - m'$.

10. If $\tan\beta = \dfrac{n \sin\alpha \cos\alpha}{1 - n\sin^2\alpha}$, shew that $\tan(\alpha - \beta) = (1-n)\tan\alpha$.

11. Two triangles stand on the same base, determine in terms of the base and of the tangents of the angles at the base the distance between the vertices of the triangles.

12. Give a construction for finding the common tangents of two circles, and shew that if through the intersection O of two of the common tangents which meet in the line joining the centres of the two circles, there be drawn a transversal meeting the circles in A, A' and B, B' respectively, then (the points denoted by B, B' being properly chosen) $OA.OB' = OA'.OB$ is independent of the position of the transversal.

13. Shew that a triangle made to revolve in the same

direction about its three angular points in a proper order through angles double of the angles of the triangle at the same angular points respectively will resume its original position.

14. Given a pair of conjugate diameters of a conic section, find geometrically the directions of the principal diameters, (1) in the case of the hyperbola, (2) in that of the ellipse.

15. A cone whose semivertical angle is $\tan^{-1}\dfrac{1}{\sqrt{2}}$ is enclosed in the circumscribing spherical surface; shew that it will rest in any position.

16. A string $ABCDEP$ is attached to the centre A of a pulley whose radius is r, it then passes over a fixed point B and under the pulley which it touches in the points C and D; it afterwards passes over a fixed point E and has a weight P attached to its extremity; BE is horizontal and $=\dfrac{5r}{3}$, and DE is vertical; shew that if the system be in equilibrium the weight of the pulley is $\dfrac{5P}{2}$, and find the distance AB.

17. A body of given elasticity e is projected along a horizontal plane from the middle point of one of the sides of an isosceles right-angled triangle so as after reflexion at the hypotenuse and remaining side to return to the same point; shew that the cotangents of the angles of reflexion are $e+1$ and $e+2$ respectively.

18. If a heavy body be projected in a direction inclined to the horizon, shew that the time of moving between two points at the extremities of a focal chord of the parabolic path is proportional to the product of the velocities of the body at the two points.

19. If a body describe an ellipse round a centre of force in the focus, shew that the sum of the reciprocals of the squares of the velocities at the extremities of any chord passing through the other focus is constant.

20. A hollow cone floats in a fluid with its vertex upwards and axis vertical; determine the density of the air contained in the hollow cone.

21. A sphere composed of two hemispheres of different refractive powers is placed in the path of a pencil of light in such manner that the axis of the pencil is perpendicular to the plane of junction and passes through the centre; determine the geometrical focus of the refracted pencil.

22. Altitudes of the same heavenly body are observed from the deck of a ship and from the top of the mast the height of which from the deck is known; find the dip of the horizon and the true altitude.

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